

Modeling Flow in Porous Media With Double Porosity/Permeability: Mathematical Model, Properties, and Analytical Solutions

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Geomaterials such as vuggy carbonates are known to exhibit multiple spatial scales. A common manifestation of spatial scales is the presence of (at least) two different scales of pores with different hydromechanical properties. Moreover, these pore-networks are connected through fissures and conduits. Although some models are available in the literature to describe flows in such media, they lack a strong theoretical basis. This paper aims to fill this gap in knowledge by providing the theoretical foundation for the flow of incompressible single-phase fluids in rigid porous media that exhibit double porosity/permeability. We first obtain a mathematical model by combining the theory of interacting continua and the maximization of rate of dissipation (MRD) hypothesis, and thereby provide a firm thermodynamic underpinning. The governing equations of the model are a system of elliptic partial differential equations (PDEs) under a steady-state response and a system of parabolic PDEs under a transient response. We then present, along with mathematical proofs, several important mathematical properties that the solutions to the model satisfy. We also present several canonical problems with analytical solutions which are used to gain insights into the velocity and pressure profiles, and the mass transfer across the two pore-networks. In particular, we highlight how the solutions under the double porosity/permeability differ from the corresponding ones under Darcy equations. [DOI: 10.1115/1.4040116]

1 Introduction and Motivation

Most models of flow in porous media make the simplifying assumption that the domain consists of a system of similar-sized pores connected by a single pore-network. In reality, many geomaterials such as aggregated soils or fissured rocks exhibit two or more dominant pore-scales connected by multiple pore-networks [1–7] that display significantly different hydromechanical properties such as disparate permeabilities and different orders of volume fractions. As an example, let us consider a pile consisting of large clay pieces. In such a medium, clay pieces form the macronetwork with the voids between them being considered as the macropores while the existing system of fissures and cracks form the micropores. It is worth mentioning that in such a system, the degradation of macropores over time leads to an increase in the amount of micropores. Moreover, due to recent advances in 3D printing and additive manufacturing, it is also possible to obtain synthetic media with two distinct pore-networks. For example, in Fig. 1, the pores between the spheres construct the macronetwork, while the micronetwork has been generated by drilling cylindrical holes in the spheres.

Porous materials with two dominant pore-networks have been studied in the literature under the subject of either the dual-porosity or dual-permeability. (A recent work [6] even documents prior works along with a comprehensive treatment of porous materials with *multiple* pore-networks.) However, there is a subtle difference in the phenomena the two words describe, and therefore, it is necessary to clarify what we mean by dual-porosity and dual-permeability models. Note that, “dual” and “double” have

been used equivalently in the literature, as will be done in this paper.

The main assumption in a *dual-porosity* model is that the permeability of the macropores is much greater than the permeability of the micropores, while the porosity of the former is much smaller than the porosity of the latter. In other words, fluid is mostly trapped within the micropores, while macropores form the major fluid pathways due to their higher permeability. Hence, the liquid phase is divided into mobile and immobile regions with the possibility of fluid exchange between them [8–10]. In the most general case, many rocks contain an independent system of fractures superimposed on the porous matrix and are commonly known as an intermediate porous medium. Such media are typically idealized using the dual-porosity model which is of high interest in petroleum reservoirs. Samples of such rocks are limestones or dolomites. However, other sedimentary rocks such as cherty shale or siltstone also exhibit the same characteristics [11]. Carbonate rocks have been known to exhibit macropores in form of fractures and joints for a long time, and the importance of such porosities in the sandstones was emphasized on by Hayes [12] and Schmidt and McDonald [13]. Moreover, studies have revealed that natural soils, especially the compacted ones, have two levels of structure, leading to the appearance of two main classes of the

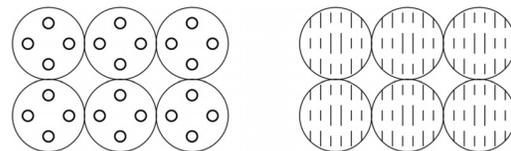


Fig. 1 Conceptualization of a synthetic medium with two pore-scales

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pores (macro- and micropores) corresponding to the two levels of soil structure [14]. Soils exhibiting such a division of pores can also be idealized using the concept of dual-porosity. The first dual-porosity model is commonly attributed to Barenblatt et al. and addressed the flow through a *fractured* porous medium [15]. In this paper, a term has been introduced based on the dimensional analysis arguments to account for the mass transfer across the two pore-scales (i.e., matrix pores and fissures). Warren and Root [11] later introduced two parameters for characterizing dual-porosity media; one measures the fluid capacitance in the macropores, and the other accounts for the interporosity flow. Dykhuizen [16] proposed a new nonlinear coupling term for double porosity based on the models proposed by Barenblatt et al. [15] and Warren and Root [11]. This dual-porosity model, unlike the previous ones, accounts for the diffusion across pore-networks and is valid even for unsteady conditions.

In contrast to dual-porosity models, the term *dual-permeability* pertains to the case where the fluid flows through both micropores and macropores, and there can be mass transfer across the pore-networks [17,18]. Different approaches have been used to describe flow and transport using dual-permeability models. In some cases, the flow in both micropores and macropores has been described using similar governing equations, while in others, different formulations have been considered in the two pore-networks [9]. However, most of the works on dual-permeability have considered the macronetwork to be *fractures* with much higher permeability than the micronetwork.

Herein, we generalize by assuming that there are two pore-networks with their own porosity and permeability, and there is a mass transfer across the pore-networks. The macronetwork can be a network of fractures, or can be another pore-network. It is possible to identify the presence of multiple pore sizes using experimental techniques such as the Brunauer–Emmett–Teller method [19]. Moreover, the multiple pore-networks can be characterized using modern techniques like μ -CT [20]. We shall refer to the aforementioned general treatment as the *double porosity/permeability model*. Figure 2 represents the fractured porous medium idealized by dual-porosity model as well as a porous medium with two pore-networks idealized by a double porosity/permeability model. The vertical and horizontal arrows represent the fluid pathways and the mass transfer within the domain. In the fractured porous medium idealized by dual-porosity model, mass transfer can occur between matrix pores and the fractures, and the fluid mostly passes through the fissures due to their higher permeability. In the porous medium with two pore-networks, mass is transferred across the two pore-networks but in this case, both micropores and macropores provide the pathways for pore fluid.

Although various models have been developed for double porosity/permeability over the years, many of them are applicable to simple settings and are valid only under stringent conditions. Some mathematically oriented works derived dual-porosity models using the theory of mathematical homogenization (e.g., see Refs. [21–23]). However, these papers did not address the relevant thermomechanical underpinning, and did not provide a coherent framework that makes it possible to obtain generalizations of those models in such a way that the thermomechanics principles are satisfied. Homogenization is a mathematical tool for *up-scaling* differential equations. In homogenization theory, a complex, rapidly varying medium is represented by a slowly varying medium in which the fine-scale structure is averaged out properly, and a “homogenized” or “effective” system of equations is obtained at the macroscopic level [24]. In other words, the problem at hand is embedded in a set of problems which are parametrized by a scaling parameter [25]. Most importantly, the presentations of most of the prior works on double porosity/permeability seem rather ad hoc, especially with respect to the treatment of mass transfer across the pore-networks. A notable exception is Ref. [5], which is discussed later. The ad hoc treatment of mass transfer is one of the main hurdles researchers are faced with while generalizing the mathematical model to more

complicated situations like multiphase flows and considering the effect of deformation of the porous solid along with flow in multiple pore-networks. Herein, we put the double porosity/permeability model under a firm footing with strong thermodynamic and mathematical underpinnings. In particular, we give a firm basis for the mass transfer across the pore-networks, and a mathematical framework amenable to further generalizations of the model.

The basic philosophy in our modeling approach can be stated as follows: (a) there exist (at least) two different pore-networks; (b) each pore-network is assumed to be a continuum, and transport of mass and chemical species can occur within each pore-network; (c) mass can be transferred between the pore-networks. The parameters and quantities in the model represent values that are averaged over a representative volume element whose existence is either tacitly or explicitly assumed in most of the double porosity/permeability models. For simplicity, we will model the flow in both networks using similar governing equations (i.e., Darcy-type equations), but one can use different descriptions of flows in different pore-networks.

An important paper toward the theoretical development of double porosity models is by Borja and Koliji [5], which, similar to our paper, employed the theory of interacting continua. However, our paper and Ref. [5] are complementary at many levels, and there are important differences in these two approaches, as discussed below:

- (i) *Deformation of the porous solid*: Borja and Koliji have considered deformation of the solid, and specifically, they have assumed the porous solid to be elastoplastic. On the other hand, we have assumed the porous solid to be rigid.

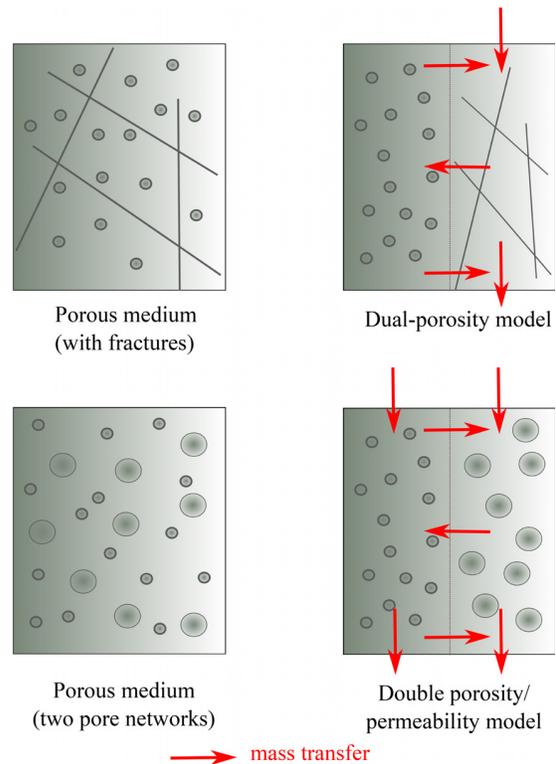


Fig. 2 Porous media and their idealizations: top part of the figure displays the idealization of a fractured porous medium using the dual-porosity model and the bottom part shows the idealization of a porous medium with two distinct pore-networks using the double porosity/permeability model. The arrows represent the fluid pathways and the mass transfer within the domain.

(ii) *Dissipation*: The theoretical framework in Ref. [5] utilizes the maximum plastic dissipation hypothesis (which is a popular approach to derive plasticity models; for example, see Refs. [26] and [27]). On the other hand, we have employed the maximization of rate of dissipation (MRD) hypothesis.

In Ref. [5], constitutive equations (e.g., hardening rule) are derived by taking into account the dissipation due to plastic deformation in the porous solid.

On the other hand, our constitutive equations are derived by considering the internal dissipation due to fluid flow in each pore-network and in the connectors. In this process, the governing equation for mass transfer across the pore-networks directly stems from the MRD hypothesis, which is one of the highlights of our paper.

- (iii) *Effective stress principle*: By appealing to the maximum plastic dissipation, Borja and Koliji obtain the effective stress for the porous solid with double porosity; which is an important theoretical advancement toward the modeling of deformable porous media. Since we have assumed the porous solid to be rigid, a discussion on stresses in the porous solid does not arise.
- (iv) *Compressibility versus incompressibility*: Borja and Koliji have assumed the fluid to be compressible. On the other hand, we have assumed the fluid to be incompressible. Their approach is best suited for modeling migration of gases from micropore network to macropore network. Our model is more suitable for modeling flow of liquids in porous media with two distinct pore-networks.
- (v) *Interpretation of pressure*: Due to the above difference (in terms of (in)compressibility) the interpretation of pressure in these two works is different. In our paper, the pressure in a pore-network is a Lagrange multiplier to enforce the incompressibility of the fluid in that pore-network. In their work, the pressure is a thermodynamic pressure, and hence, one needs to specify the so-called equation of state (which is a constitutive specification).

The rest of this paper is organized as follows. Section 2 outlines the governing equations for a double porosity/permeability model. Section 3 presents a mathematical framework for deriving porous media models by appealing to the theory of interacting continua and the maximization of rate of dissipation hypothesis, and obtains the double porosity/permeability model as a special case. Several mathematical properties of this model are derived in Sec. 4. An analytical solution procedure is presented in Sec. 5. Several canonical problems along with their analytical solutions are given in Sec. 6. Finally, conclusions are drawn in Sec. 7.

Throughout this paper, repeated indices do not imply summation.

2 Mathematical Model

Consider a bounded domain $\Omega \subset \mathbb{R}^{\text{nd}}$, where “nd” denotes the number of spatial dimensions. The boundary $\partial\Omega$ is assumed to be piecewise smooth. Mathematically, $\partial\Omega := \text{cl}(\Omega) - \Omega$, where $\text{cl}(\cdot)$ denotes the set closure [28]. A spatial point in Ω is denoted by \mathbf{x} . The gradient and divergence operators with respect to \mathbf{x} are, respectively, denoted by $\text{grad}[\cdot]$ and $\text{div}[\cdot]$. The unit outward normal to the boundary is denoted by $\hat{\mathbf{n}}(\mathbf{x})$.

We are interested in studying the flow of an incompressible single-phase fluid in a rigid porous medium that consists of two distinct pore-networks. These pore-networks are connected by conduits and/or fissures, and hence, there can be mass transfer across the pore-networks. We shall refer to these two pore-networks as macropore and micropore networks, and identify them using subscripts 1 and 2, respectively. The permeability tensors for these pore-networks are denoted by $\mathbf{K}_1(\mathbf{x})$ and $\mathbf{K}_2(\mathbf{x})$, which are assumed to be anisotropic and spatially inhomogeneous second-order tensors. The porosities in these pore-networks are

denoted by $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$. *Strictly speaking, these two parameters should be referred to as volume fractions.* The true density and the coefficient of viscosity of the fluid are denoted by γ and μ , respectively. The bulk densities in the macropores and micropores are, respectively, denoted by $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{x})$. That is,

$$\rho_1(\mathbf{x}) = \phi_1(\mathbf{x})\gamma \quad \text{and} \quad \rho_2(\mathbf{x}) = \phi_2(\mathbf{x})\gamma \quad (1)$$

The pressure scalar fields in the macropore and micropore networks are, respectively, denoted by $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$. The true (or seepage) velocity vector fields in the two pore-networks are denoted by $\mathbf{v}_1(\mathbf{x})$ and $\mathbf{v}_2(\mathbf{x})$, which denote the rate of discharge of fluid per unit cross-sectional area of pores [29]. The discharge (or Darcy) velocities, $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$ (which denote the discharge of fluid per unit of the *total* cross-sectional area) are related to the true velocities as follows:

$$\mathbf{u}_1(\mathbf{x}) = \phi_1(\mathbf{x})\mathbf{v}_1(\mathbf{x}) \quad \text{and} \quad \mathbf{u}_2(\mathbf{x}) = \phi_2(\mathbf{x})\mathbf{v}_2(\mathbf{x}) \quad (2)$$

Equation (2) further implies that the true velocity $\mathbf{v}_i(\mathbf{x})$ is always greater than the discharge velocity $\mathbf{u}_i(\mathbf{x})$.

For the macropore network, we shall decompose the boundary into two parts: Γ_1^v and Γ_1^p . Γ_1^v denotes the part of the boundary on which the normal component of the velocity in the macropore network is prescribed. Γ_1^p is that part of the boundary on which the pressure in the macropore network is prescribed. Likewise, for the micropore network, the boundary is decomposed into two parts: Γ_2^v and Γ_2^p . For mathematical well-posedness, we assume that

$$\begin{aligned} \Gamma_1^v \cup \Gamma_1^p &= \partial\Omega \quad \text{and} \quad \Gamma_1^v \cap \Gamma_1^p = \emptyset \\ \Gamma_2^v \cup \Gamma_2^p &= \partial\Omega \quad \text{and} \quad \Gamma_2^v \cap \Gamma_2^p = \emptyset \end{aligned} \quad (3)$$

The governing equations in terms of the true velocities can be written as follows:

$$\mu\phi_1^2\mathbf{K}_1^{-1}\mathbf{v}_1(\mathbf{x}) + \phi_1\text{grad}[p_1] = \rho_1\mathbf{b}(\mathbf{x}) \quad \text{in } \Omega \quad (4a)$$

$$\mu\phi_2^2\mathbf{K}_2^{-1}\mathbf{v}_2(\mathbf{x}) + \phi_2\text{grad}[p_2] = \rho_2\mathbf{b}(\mathbf{x}) \quad \text{in } \Omega \quad (4b)$$

$$\text{div}[\phi_1\mathbf{v}_1] = +\chi(\mathbf{x}) \quad \text{in } \Omega \quad (4c)$$

$$\text{div}[\phi_2\mathbf{v}_2] = -\chi(\mathbf{x}) \quad \text{in } \Omega \quad (4d)$$

$$\mathbf{v}_1(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = v_{n1}(\mathbf{x}) \quad \text{on } \Gamma_1^v \quad (4e)$$

$$\mathbf{v}_2(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = v_{n2}(\mathbf{x}) \quad \text{on } \Gamma_2^v \quad (4f)$$

$$p_1(\mathbf{x}) = p_{01}(\mathbf{x}) \quad \text{on } \Gamma_1^p \quad (4g)$$

$$p_2(\mathbf{x}) = p_{02}(\mathbf{x}) \quad \text{on } \Gamma_2^p \quad (4h)$$

where $\mathbf{b}(\mathbf{x})$ is the specific body force. $v_{n1}(\mathbf{x})$ is the prescribed normal component of the velocity on the boundary in the macropores, and $v_{n2}(\mathbf{x})$ is the prescribed normal component of the velocity on the boundary in the micropores. $p_{01}(\mathbf{x})$ is the prescribed pressure on the boundary in the macropores, and $p_{02}(\mathbf{x})$ is the prescribed pressure on the boundary in the micropores. $\chi(\mathbf{x})$ is the rate of volume of the fluid that is exchanged between the two pore-networks per unit volume of the porous medium. *In the rest of the paper, $\chi(\mathbf{x})$ is simply referred to as the mass transfer.* Herein, the mass transfer is modeled as follows:

$$\chi(\mathbf{x}) = -\frac{\beta}{\mu}(p_1(\mathbf{x}) - p_2(\mathbf{x})) \quad (5)$$

where β is a dimensionless characteristic of the porous medium. The above expression for the mass transfer can be traced back to Barenblatt et al. [15], which was derived based on a dimensional analysis argument. Some works in the literature refer to such an

expression for the interporosity flow as the “Barenblatt-Zheltoev” model, for example, see Ref. [30]. Tables are provided in the literature for β (e.g., see Ref. [15]). Also, with the modern techniques like micro-CT, one can estimate β using multiscale methods by considering the actual pore-networks. This is an active area of research. To provide a physical insight into β , consider that the two pore-networks are connected by conduits with radius R and length L . Then, $\beta = R^2/(8L^2)$. If the two pore-networks are connected by fissures, which can be idealized as parallel plates with length L and separated by a width of h , then $\beta = h^2/(12L^2)$. These expressions are obtained by assuming Poiseuille flow in conduits and Couette flow in fissures. In reality, the two pore-networks can be connected by both conduits and fissures, and these connectors can even be tortuous.

Equation (5) represents a linear mass transfer model and is one of the simplest models available in the literature. Under this model, it is assumed that the fluid can be exchanged between the two pore-networks if there exists a sufficiently smooth change of pressure between the networks. Although such an interporosity flow equation seems simple, it has been proven to maintain the essential features of flow through the naturally fractured reservoirs [30]. Alternatively, multirate mass transfer (MRMT) models have been proposed and studied in the literature; for example, see Refs. [10] and [31]. MRMT models employ a Fickian-type transport model (as opposed to a flow model) for the mass transfer from the micropore network to the macropore network. Thus, the resulting problem under a MRMT model will be a coupled flow-transport, as opposed to pure flow problem under the linear mass transfer model. Due to a transport model for interpore mass transfer, MRMT models are particularly useful to study the migration of gases in porous media with multiple pore-networks. On the other hand, the linear mass transfer model will be quite useful to study the flow of liquids in such porous media.

A comparison (in terms of theoretical implications and predictive capabilities) of the linear mass transfer model employed in this paper and MRMT models is worthy of a systematic scientific inquiry. But such a study is beyond the scope of this paper.

An alternate form of the governing equations, which is particularly convenient for numerical formulations, is written as follows in terms of discharge velocities:

$$\mu \mathbf{K}_1^{-1} \mathbf{u}_1(\mathbf{x}) + \text{grad}[p_1] = \gamma \mathbf{b}(\mathbf{x}) \quad \text{in } \Omega \quad (6a)$$

$$\mu \mathbf{K}_2^{-1} \mathbf{u}_2(\mathbf{x}) + \text{grad}[p_2] = \gamma \mathbf{b}(\mathbf{x}) \quad \text{in } \Omega \quad (6b)$$

$$\text{div}[\mathbf{u}_1] = +\chi(\mathbf{x}) \quad \text{in } \Omega \quad (6c)$$

$$\text{div}[\mathbf{u}_2] = -\chi(\mathbf{x}) \quad \text{in } \Omega \quad (6d)$$

$$\mathbf{u}_1(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = \phi_1 v_{n1}(\mathbf{x}) =: u_{n1}(\mathbf{x}) \quad \text{on } \Gamma_1^v \quad (6e)$$

$$\mathbf{u}_2(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = \phi_2 v_{n2}(\mathbf{x}) =: u_{n2}(\mathbf{x}) \quad \text{on } \Gamma_2^v \quad (6f)$$

$$p_1(\mathbf{x}) = p_{01}(\mathbf{x}) \quad \text{on } \Gamma_1^p \quad (6g)$$

$$p_2(\mathbf{x}) = p_{02}(\mathbf{x}) \quad \text{on } \Gamma_2^p \quad (6h)$$

The above governing equations, which are for a steady-state response, form a system of elliptic partial differential equations (PDEs). It needs to be emphasized that many of the mathematical properties presented later in this paper depend on the linearity of the mass transfer model (i.e., Eq. (5)).

3 Proposed Approach to Develop Double Porosity/ Permeability Models

Several porous media models have been developed using the theory of interacting continua for flow, reactive-transport, and/or

deformation of multiple constituents in a single pore-network or multiple ones by treating each component to be either a fluid, a solid or a chemical species. These works include [5,32–35], just to name a few. The maximization of rate of dissipation hypothesis, which is also referred to as the orthogonality principle and is similar in spirit to the maximization of entropy production, has been first proposed by Ziegler to derive the constitutive relations [36]. An attractive feature of this hypothesis is that prescription of two physically meaningful functionals (Helmholtz potential and dissipation functional) provides the constitutive relations even for a phenomenon which involves a multitude of interacting processes [37]. Subsequently, this hypothesis has been successfully employed to develop constitutive models for a wide variety of physical phenomena, which include inelasticity [38], anisotropic fluids [39], degradation of materials [40], and diffusion in viscoelastic polymers [41]. However, the combination of this hypothesis with the theory of interacting continua has not been previously utilized to derive constitutive relations for porous media with multiple pore-networks.

Thus, one of the goals of this paper is to combine the theory of interacting continua and the maximization of rate of dissipation hypothesis for obtaining a coherent framework to derive models of flow in porous media with multiple pore-networks. Specifically, we consider the flow of an incompressible single-phase fluid in a rigid porous domain.

3.1 Theory of Interacting Continua: A General Setting.

The porous medium is treated as a mixture of \mathcal{N} constituents. We use the word “constituent” to refer to the porous solid or a pore-network. This usage is slightly different from the usual mixture theory models. In a typical mixture theory model, a constituent refers to a different physical/chemical component or a different phase.

We denote the bulk density, specific body force, partial Cauchy stress, specific internal energy, specific Helmholtz potential, temperature, heat flux vector, and specific entropy of the i -th constituent by ρ_i , \mathbf{b}_i , \mathbf{T}_i , U_i , A_i , θ_i , \mathbf{q}_i , and η_i ; respectively.

3.1.1 Kinematics. We denote the time by t . Under the theory of interacting continua, a mixture is treated as a superposition of multiple continua each following its own motion. At a given instance of time, each spatial point \mathbf{x} in the mixture is occupied simultaneously by \mathcal{N} different particles \mathbf{p}_i ($i = 1, \dots, \mathcal{N}$), one from each constituent. The motion of the constituents can be written as

$$\mathbf{x} = \phi_i(\mathbf{p}_i, t) \quad i = 1, \dots, \mathcal{N} \quad (7)$$

with the corresponding velocities defined as follows:

$$\mathbf{v}_i = \frac{\partial \phi_i(\mathbf{p}_i, t)}{\partial t} \quad (8)$$

\mathbf{p}_i in previous equations represents the material points in the reference (undeformed) configuration which is a vector-valued parameter.

The gradient of motion of the i -th constituent is denoted by \mathbf{F}_i . That is,

$$\mathbf{F}_i = \frac{\partial \mathbf{x}}{\partial \mathbf{p}_i} \quad (9)$$

Let

$$\mathbf{L}_i := \text{grad}[\mathbf{v}_i] \quad (10)$$

and

$$\mathbf{D}_i := \text{sym}[\mathbf{L}_i] = \frac{1}{2} \left(\text{grad}[\mathbf{v}_i] + \text{grad}[\mathbf{v}_i]^T \right) \quad (11)$$

We introduce the following material time derivative defined on arbitrary scalar field ψ and vector field \mathbf{w} :

$$\frac{D^{(i)}\psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}_i \cdot \text{grad}[\psi] \quad (12)$$

and

$$\frac{D^{(i)}\mathbf{w}}{Dt} = \frac{\partial\mathbf{w}}{\partial t} + \text{grad}[\mathbf{w}]\mathbf{v}_i \quad (13)$$

It is important to note that the material derivative $D^{(i)}(\cdot)/Dt$ follows the motion of the i -th constituent.

3.1.2 Balance Laws. The local form of the *balance of mass* of the i -th constituent can be written as follows:

$$\frac{\partial\rho_i}{\partial t} + \text{div}[\rho_i\mathbf{v}_i] = m_i \quad (14)$$

where m_i is the rate of mass transfer into the i -th pore-network per unit volume of the porous medium. The local form of the overall balance of mass for the porous medium takes the following form:

$$\sum_{i=1}^{\mathcal{N}} m_i = 0 \quad (15)$$

Under the theory of interacting continua, the mechanical interaction between constituents is modeled using interaction terms [35]. Herein, we denote the interaction term for the i -th constituent due to the presence of other constituents by \mathbf{i}_i . The *balance of linear momentum* of the i -th constituent, by taking into account the balance of mass (i.e., Eq. (14)), takes the following form:

$$\rho_i \frac{D^{(i)}\mathbf{v}_i}{Dt} = \text{div}[\mathbf{T}_i] + \rho_i\mathbf{b}_i + \mathbf{i}_i \quad (16)$$

The local form of the overall balance of linear momentum for the porous medium takes the following form:

$$\sum_{i=1}^{\mathcal{N}} \mathbf{i}_i = 0 \quad (17)$$

We assume a stronger version of the *balance of angular momentum* for each constituent by asserting that

$$\mathbf{T}_i = \mathbf{T}_i^T \quad \forall i = 1, \dots, \mathcal{N} \quad (18)$$

The *balance of energy* of the i -th constituent, by taking into account the balance of mass (i.e., Eq. (14)) and the balance of the linear momentum (i.e., Eq. (16)), takes the following form:

$$\rho_i \frac{D^{(i)}U_i}{Dt} = \mathbf{T}_i \cdot \mathbf{L}_i - \text{div}[\mathbf{q}_i] + \rho_i r_i + \varepsilon_i^s \quad (19)$$

where ε_i^s is energy supply to the i -th constituent due to the interaction with other constituents, and r_i is the (external) specific heat supply to the i -th constituent. The local form of the overall balance of energy for the porous media takes the following form:

$$\sum_{i=1}^{\mathcal{N}} (\varepsilon_i^s + \mathbf{i}_i \cdot \mathbf{v}_i) = 0 \quad (20)$$

The *second law of thermodynamics*, which is a global law, is written as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{i=1}^{\mathcal{N}} \int_{\Omega} \rho_i \eta_i d\Omega + \sum_{i=1}^{\mathcal{N}} \int_{\partial\Omega} \rho_i \eta_i \mathbf{v}_i \cdot \hat{\mathbf{n}} d\Gamma \geq & - \sum_{i=1}^{\mathcal{N}} \int_{\partial\Omega} \frac{\mathbf{q}_i \cdot \hat{\mathbf{n}}}{\theta_i} d\Gamma \\ & + \sum_{i=1}^{\mathcal{N}} \int_{\Omega} \frac{\rho_i r_i}{\theta_i} d\Omega + \sum_{i=1}^{\mathcal{N}} \int_{\Omega} m_i \eta_i d\Omega \end{aligned} \quad (21)$$

Recall that $\hat{\mathbf{n}}$ denotes the outward normal to the boundary. The above inequality can be considered as an extension of the Clausius–Duhem inequality to multiconstituent media. We assume the local form to hold, which is stronger than the second law of thermodynamics. The local form corresponding to the above inequality reads

$$\frac{\partial}{\partial t} \sum_{i=1}^{\mathcal{N}} \rho_i \eta_i + \sum_{i=1}^{\mathcal{N}} \text{div}[\rho_i \eta_i \mathbf{v}_i] \geq \sum_{i=1}^{\mathcal{N}} \left(-\text{div} \left[\frac{\mathbf{q}_i}{\theta_i} \right] + \frac{\rho_i r_i}{\theta_i} + m_i \eta_i \right) \quad (22)$$

Using the balance of mass (i.e., Eq. (14)), the above inequality can be simplified as follows:

$$\sum_{i=1}^{\mathcal{N}} \left(\rho_i \frac{D^{(i)}\eta_i}{Dt} + \text{div} \left[\frac{\mathbf{q}_i}{\theta_i} \right] - \frac{\rho_i r_i}{\theta_i} \right) \geq 0 \quad (23)$$

By dividing both sides of Eq. (19) by θ_i , summing over the number of constituents, and subtracting the result from the inequality (23), we obtain the following inequality:

$$\sum_{i=1}^{\mathcal{N}} \rho_i \left(\frac{D^{(i)}\eta_i}{Dt} - \frac{1}{\theta_i} \frac{D^{(i)}U_i}{Dt} \right) \geq \sum_{i=1}^{\mathcal{N}} \frac{1}{\theta_i} \left(-\mathbf{T}_i \cdot \mathbf{L}_i + \frac{1}{\theta_i} \mathbf{q}_i \cdot \text{grad}[\theta_i] - \varepsilon_i^s \right) \quad (24)$$

We now replace the specific internal energy with the specific Helmholtz potential using a Legendre transformation, which can be mathematically written as follows:

$$U_i = A_i + \theta_i \eta_i \quad \text{with} \quad \eta_i = -\frac{\partial A_i}{\partial \theta_i} \quad (25)$$

We assume the functional dependence of the specific Helmholtz potential to be $A_i = A_i(\mathbf{F}_i, \theta_i)$. Noting that

$$\frac{D^{(i)}\mathbf{F}_i}{Dt} = \mathbf{L}_i \mathbf{F}_i \quad (26)$$

and using Eq. (20), inequality (24) can be written as follows:

$$\sum_{i=1}^{\mathcal{N}} \frac{1}{\theta_i} \left(\rho_i \frac{\partial A_i}{\partial \mathbf{F}_i} \mathbf{F}_i^T - \mathbf{T}_i \right) \cdot \mathbf{L}_i + \sum_{i=1}^{\mathcal{N}} \frac{1}{\theta_i} \left(\frac{1}{\theta_i} \mathbf{q}_i \cdot \text{grad}[\theta_i] + \mathbf{i}_i \cdot \mathbf{v}_i \right) \leq 0 \quad (27)$$

The above inequality can be converted into a convenient equality by introducing a non-negative functional, $\Psi \geq 0$, and the resulting equality reads

$$\sum_{i=1}^{\mathcal{N}} \frac{1}{\theta_i} \left(\rho_i \frac{\partial A_i}{\partial \mathbf{F}_i} \mathbf{F}_i^T - \mathbf{T}_i \right) \cdot \mathbf{L}_i + \sum_{i=1}^{\mathcal{N}} \frac{1}{\theta_i} \left(\frac{1}{\theta_i} \mathbf{q}_i \cdot \text{grad}[\theta_i] + \mathbf{i}_i \cdot \mathbf{v}_i \right) + \Psi = 0 \quad (28)$$

where Ψ is the rate of entropy production per unit volume. The above equation is referred to as the *reduced energy-entropy equation*. If all the constituents have the same temperature, $\theta_i = \theta$, (i.e.,

the mixture is in thermal equilibrium) then, the above equation can be written as follows:

$$\sum_{i=1}^{\mathcal{N}} \left(\rho_i \frac{\partial A_i}{\partial \mathbf{F}_i} \mathbf{F}_i^T - \mathbf{T}_i \right) \cdot \mathbf{L}_i + \sum_{i=1}^{\mathcal{N}} \left(\frac{1}{\theta} \mathbf{q}_i \cdot \text{grad}[\theta] + \mathbf{i}_i \cdot \mathbf{v}_i \right) + \zeta = 0 \quad (29)$$

where the rate of dissipation per unit volume is defined as follows:

$$\zeta = \theta \Psi \quad (30)$$

Since $\Psi \geq 0$ and $\theta > 0$, $\zeta \geq 0$. Equation (29) is referred to as the *reduced energy-dissipation equation* for multiconstituent media. It should be emphasized that the mixture is in thermal equilibrium (i.e., all the constituents have the same temperature at a spatial point) which does *not* mean the process is isothermal.

3.2 A Simplified Framework for Double Porosity/Permeability Models. The above framework is presented in a general setting. We now provide a simplified framework for the problem at hand, which pertains to the modeling of creeping flows (i.e., flows in which convective inertial effects can be neglected) of an incompressible fluid in rigid porous media with two pore-networks. To this end, the following choices are made:

- (i) There are two pore-networks and a rigid porous medium. Strictly speaking, there are three constituents. Since the porous solid is rigid, its motion will be neglected, and all the balance laws for this constituent are assumed to be trivially satisfied. Hence, one can take $\mathcal{N} = 2$.
- (ii) All constituents at a spatial point have the same temperature (i.e., $\theta_i(\mathbf{x}, t) = \theta(\mathbf{x}, t)$). However, this temperature can vary with spatial position and time.
- (iii) There is no heat transfer. That is, $\mathbf{q}_i = \mathbf{0}$ and $r_i = 0$.
- (iv) We assume that the porosities do not change with time. This is acceptable, as the porous solid is assumed to be rigid. That is,

$$\frac{\partial \phi_i}{\partial t} = 0 \quad (i = 1, 2) \quad (31)$$

- (v) The fluid in each pore-network is incompressible, which, mathematically, translates into the following equations:

$$\frac{D^{(i)}\gamma}{Dt} \equiv \frac{\partial \gamma}{\partial t} + \mathbf{v}_i \cdot \text{grad}[\gamma] = 0 \quad (i = 1, 2) \quad (32)$$

Noting the above relation, Eq. (31) and the balance of the mass for the mixture (i.e., Eq. (15)) imply that the balance of mass for an incompressible fluid in each pore-network can be written as follows:

$$\text{div}[\phi_1 \mathbf{v}_1] = +\chi \quad \text{and} \quad \text{div}[\phi_2 \mathbf{v}_2] = -\chi \quad (33)$$

where $\chi = m_1/\gamma = -m_2/\gamma$ accounts for the mass transfer from the macropore network to the micropore network. Note that these incompressibility constraints remain the same in both transient and steady-state responses.

- (vi) The velocity in each pore-network and its (spatial) gradient are assumed to be small so that the term “grad[\mathbf{v}_i] \mathbf{v}_i ” can be neglected. That is, convective inertial effects can be neglected in each pore-network. Then, the balance of linear momentum in each pore-network for a transient response reads

$$\rho_1 \frac{\partial \mathbf{v}_1}{\partial t} = \text{div}[\mathbf{T}_1] + \rho_1 \mathbf{b}(\mathbf{x}) + \mathbf{i}_1 \quad (34)$$

and

$$\rho_2 \frac{\partial \mathbf{v}_2}{\partial t} = \text{div}[\mathbf{T}_2] + \rho_2 \mathbf{b}(\mathbf{x}) + \mathbf{i}_2 \quad (35)$$

and the corresponding ones in a steady-state response reads

$$\text{div}[\mathbf{T}_1] + \rho_1 \mathbf{b}(\mathbf{x}) + \mathbf{i}_1 = 0 \quad (36)$$

and

$$\text{div}[\mathbf{T}_2] + \rho_2 \mathbf{b}(\mathbf{x}) + \mathbf{i}_2 = 0 \quad (37)$$

Note that the balance of linear momentum for the mixture (i.e., Eq. (17)) does *not* imply that the interaction terms of both the pore-networks add up to zero. One should not forget about the porous solid. Although we have assumed the porous solid to be rigid and have not documented the balance laws pertaining to it, it does have an interaction term. The sum of all the three interaction terms (one for each pore-network and one for the porous solid) should add up to zero, which is according to the balance of linear momentum for the mixture.

- (vii) We assume that the specific Helmholtz potentials satisfy the frame-indifference [42]. This will imply that the tensor $\rho_i(\partial A_i/\partial \mathbf{F}_i) \mathbf{F}_i^T$ is symmetric. The balance of angular momentum for each constituent implies that the partial Cauchy stress tensor, \mathbf{T}_i , is symmetric. The symmetry of these tensors implies that the reduced energy-dissipation can be written as follows:

$$-\left(\mathbf{T}_1 - \rho_1 \frac{\partial A_1}{\partial \mathbf{F}_1} \mathbf{F}_1^T \right) \cdot \mathbf{D}_1 - \left(\mathbf{T}_2 - \rho_2 \frac{\partial A_2}{\partial \mathbf{F}_2} \mathbf{F}_2^T \right) \cdot \mathbf{D}_2 + \mathbf{i}_1 \cdot \mathbf{v}_1 + \mathbf{i}_2 \cdot \mathbf{v}_2 + \zeta = 0 \quad (38)$$

We now obtain the constitutive relations for the Cauchy stresses, the interaction terms, and the mass transfer across the pore-networks using the maximization of rate of dissipation hypothesis.

3.2.1 Obtaining Constitutive Relations Using Maximization of Rate of Dissipation. We handle the mass transfer across the pore-networks using an internal variable, which will be taken as follows:

$$\int_0^t \chi(\mathbf{x}, \tau) d\tau \quad (39)$$

where τ is a dummy variable. Then, the rate of the chosen internal variable will be

$$\frac{d}{dt} \int_0^t \chi(\mathbf{x}, \tau) d\tau = \chi(\mathbf{x}, t) \quad (40)$$

The mathematical statement of the maximization of rate of dissipation hypothesis for multiconstituent media can be written as follows:

$$\underset{\mathbf{D}_1, \mathbf{D}_2, \mathbf{v}_1, \mathbf{v}_2, \chi}{\text{maximize}} \quad \zeta = \hat{\zeta}(\mathbf{D}_1, \mathbf{D}_2, \mathbf{v}_1, \mathbf{v}_2, \chi) \quad (41a)$$

$$\text{subject to} \quad -\left(\mathbf{T}_1 - \rho_1 \frac{\partial A_1}{\partial \mathbf{F}_1} \mathbf{F}_1^T \right) \cdot \mathbf{D}_1 - \left(\mathbf{T}_2 - \rho_2 \frac{\partial A_2}{\partial \mathbf{F}_2} \mathbf{F}_2^T \right) \cdot \mathbf{D}_2 + \mathbf{i}_1 \cdot \mathbf{v}_1 + \mathbf{i}_2 \cdot \mathbf{v}_2 + \zeta = 0 \quad (41b)$$

$$\phi_1 \text{tr}[\mathbf{D}_1] + \mathbf{v}_1 \cdot \text{grad}[\phi_1] = +\chi \quad (41c)$$

$$\phi_2 \text{tr}[\mathbf{D}_2] + \mathbf{v}_2 \cdot \text{grad}[\phi_2] = -\chi \quad (41d)$$

Equation (41b) is the reduced energy-dissipation equation for a two pore-network porous medium, and Eqs. (41c) and (41d) are, respectively, the incompressibility constraints for the macro- and

micro-pore-networks. Using the Lagrange multiplier method, one can rewrite the above constrained optimization problem as the following unconstrained optimization problem:

$$\begin{aligned} & \text{extremize} \\ & \mathbf{D}_1, \mathbf{D}_2, \mathbf{v}_1, \mathbf{v}_2, \zeta, p_1, p_2, \lambda \\ & \zeta + p_1 (\phi_1 \text{tr}[\mathbf{D}_1] + \mathbf{v}_1 \cdot \text{grad}[\phi_1] - \chi) \\ & + p_2 (\phi_2 \text{tr}[\mathbf{D}_2] + \mathbf{v}_2 \cdot \text{grad}[\phi_2] + \chi) \\ & + \lambda \left(- \left(\mathbf{T}_1 - \rho_1 \frac{\partial A_1}{\partial \mathbf{F}_1} \mathbf{F}_1^T \right) \cdot \mathbf{D}_1 \mathbf{i}_1 \cdot \mathbf{v}_1 \right. \\ & \left. - \left(\mathbf{T}_2 - \rho_2 \frac{\partial A_2}{\partial \mathbf{F}_2} \mathbf{F}_2^T \right) \cdot \mathbf{D}_2 + \mathbf{i}_2 \cdot \mathbf{v}_2 + \zeta \right) \end{aligned} \quad (42)$$

where λ is the Lagrange multiplier corresponding to the reduced energy dissipation equation (41b), and p_1 and p_2 are the Lagrange multipliers enforcing equations (41c) and (41d), respectively. The first-order optimality conditions of the above optimization problem yield

$$\mathbf{T}_1 = -\phi_1 p_1 \mathbf{I} + \rho_1 \frac{\partial A_1}{\partial \mathbf{F}_1} \mathbf{F}_1^T + \left(\frac{\lambda + 1}{\lambda} \right) \frac{\partial \zeta}{\partial \mathbf{D}_1} \quad (43a)$$

$$\mathbf{T}_2 = -\phi_2 p_2 \mathbf{I} + \rho_2 \frac{\partial A_2}{\partial \mathbf{F}_2} \mathbf{F}_2^T + \left(\frac{\lambda + 1}{\lambda} \right) \frac{\partial \zeta}{\partial \mathbf{D}_2} \quad (43b)$$

$$\mathbf{i}_1 = \text{grad}[\phi_1] p_1 - \left(\frac{\lambda + 1}{\lambda} \right) \frac{\partial \zeta}{\partial \mathbf{v}_1} \quad (43c)$$

$$\mathbf{i}_2 = \text{grad}[\phi_2] p_2 - \left(\frac{\lambda + 1}{\lambda} \right) \frac{\partial \zeta}{\partial \mathbf{v}_2} \quad (43d)$$

$$\frac{\partial \zeta}{\partial \chi} = -(p_1 - p_2) \quad (43e)$$

where \mathbf{I} denotes the second-order identity tensor. Of course, one needs to augment the aforementioned optimality conditions with the constraints given by Eqs. (41b)–(41d). Equations (43a)–(43e) provide general constitutive relations. One can obtain a specific constitutive model by specifying A_1 , A_2 , and ζ functionals, and their prescription is a constitutive specification. Moreover, if ζ is a homogeneous functional of order two with respect to its arguments, it can be shown that $\lambda = -2$.

3.2.2 A Specific Double Porosity/Permeability Model. One can obtain the double porosity/permeability model (as given in Sec. 2) by making the following choices and assumptions:

- (i) The specific Helmholtz potentials for the two fluid constituents are taken as follows:

$$A_1 = 0 \quad \text{and} \quad A_2 = 0 \quad (44)$$

That is, the two fluid constituents are purely viscous, which is the case even for the Stokes fluid and the fluid under the Darcy model. In the jargon of thermodynamics, the zero (specific) Helmholtz potential of a constituent means that one cannot extract useful work from the internal energy of the constituent [43].

- (ii) The rate of dissipation production is taken as follows:

$$\zeta = \mu \phi_1^2 \mathbf{v}_1 \cdot \mathbf{K}_1^{-1} \mathbf{v}_1 + \mu \phi_2^2 \mathbf{v}_2 \cdot \mathbf{K}_2^{-1} \mathbf{v}_2 + \zeta_{\text{MT}}(\chi) \quad (45)$$

where the first and second terms on the right-hand side of the equation, respectively, represent the rate of dissipation in the macropore and micropore networks, and ζ_{MT} accounts for the dissipation due to mass transfer across the two pore-networks.

- (iii) Assuming that the connectors are conduits or fissures, ζ_{MT} can be taken as follows:

$$\zeta_{\text{MT}}(\chi) = \frac{\mu}{\beta} \chi^2 \quad (46)$$

where β is a dimensionless characteristic parameter of the porous medium, as mentioned in Sec. 2. The logic behind the functional form in Eq. (46) will be evident by looking at the rate of dissipation in a Couette flow (i.e., flow in fissures) or a Poiseuille flow (i.e., flow in conduits); for example, see Ref. [44]. Noting the above choice for ζ_{MT} , it is easy to verify that the functional ζ given by Eq. (45) is a homogeneous functional of order two of its arguments.

By substituting the above constitutive specifications into Eqs. (43a)–(43e), one obtains the following constitutive relations:

$$\begin{aligned} \mathbf{T}_1 &= -\phi_1 p_1 \mathbf{I}, \quad \mathbf{T}_2 = -\phi_2 p_2 \mathbf{I}, \quad \mathbf{i}_1 = \mu \phi_1^2 \mathbf{K}_1^{-1} \mathbf{v}_1 \\ \mathbf{i}_2 &= \mu \phi_2^2 \mathbf{K}_2^{-1} \mathbf{v}_2, \quad \chi = -(p_1 - p_2) \end{aligned} \quad (47)$$

The above constitutive relations along with the balance of mass and the balance of linear momentum give rise to the following equations in a steady-state setting:

$$\mu \phi_1^2 \mathbf{K}_1^{-1} \mathbf{v}_1 + \phi_1 \text{grad}[p_1] = \rho_1 \mathbf{b}(\mathbf{x}) \quad (48a)$$

$$\mu \phi_2^2 \mathbf{K}_2^{-1} \mathbf{v}_2 + \phi_2 \text{grad}[p_2] = \rho_2 \mathbf{b}(\mathbf{x}) \quad (48b)$$

$$\text{div}[\phi_1 \mathbf{v}_1] = +\chi \quad (48c)$$

$$\text{div}[\phi_2 \mathbf{v}_2] = -\chi \quad (48d)$$

$$\chi = -\frac{\beta}{\mu} (p_1 - p_2) \quad (48e)$$

which are the governing equations under the double porosity/permeability model presented in Sec. 2.

In a transient setting, Eqs. (48a) and (48b), will be replaced by the following:

$$\rho_1 \frac{\partial \mathbf{v}_1}{\partial t} + \mu \phi_1^2 \mathbf{K}_1^{-1} \mathbf{v}_1 + \phi_1 \text{grad}[p_1] = \rho_1 \mathbf{b}(\mathbf{x}, t) \quad (49a)$$

$$\rho_2 \frac{\partial \mathbf{v}_2}{\partial t} + \mu \phi_2^2 \mathbf{K}_2^{-1} \mathbf{v}_2 + \phi_2 \text{grad}[p_2] = \rho_2 \mathbf{b}(\mathbf{x}, t) \quad (49b)$$

Equations (48c)–(48e) remain the same even in a transient setting. However, we need to prescribe the initial conditions for the velocity in each pore-network for the transient case. The governing equations for a transient response form a system of parabolic partial differential equations.

3.3 An Illustrative Generalization: An Extension of the Brinkman Model. The above framework offers an attractive setting for deriving porous media models in a consistent manner. In particular, it is possible to obtain generalizations of the double porosity/permeability model and include other physical processes. We now illustrate how to generalize the Brinkman model [45] to incorporate double pore-networks and the mass transfer across the pore-networks. To this end, we make the following choices for the specific Helmholtz potentials and the dissipation functional:

$$A_1 = A_2 = 0 \quad (50a)$$

$$\begin{aligned} \zeta &= \mu \phi_1^2 \mathbf{v}_1 \cdot \mathbf{K}_1^{-1} \mathbf{v}_1 + \mu \phi_1 \mathbf{D}_1 \cdot \mathbf{D}_1 + \mu \phi_2^2 \mathbf{v}_2 \cdot \mathbf{K}_2^{-1} \mathbf{v}_2 \\ &+ \mu \phi_2 \mathbf{D}_2 \cdot \mathbf{D}_2 + \zeta_{\text{MT}} \end{aligned} \quad (50b)$$

$$\zeta_{\text{MT}} = \frac{\mu}{\beta} \chi^2 \quad (50c)$$

A physical justification of the above choice for ζ is as follows: the first term models the dissipation due to friction at the interface of

the porous solid and the fluid in the macropore network [46]. The second term corresponds to the dissipation due to friction in the internal layers of the fluid in the macropore network. The third and fourth terms model the corresponding phenomena in the micropore network. The fifth term models the dissipation due to mass transfer in the connectors. The above choices give rise to the following constitutive relations:

$$\begin{aligned} \mathbf{T}_1 &= -\phi_1 p_1 \mathbf{I} + 2\mu\phi_1 \mathbf{D}_1, & \mathbf{i}_1 &= \mu\phi_1^2 \mathbf{K}_1^{-1} \mathbf{v}_1 \\ \mathbf{T}_2 &= -\phi_2 p_2 \mathbf{I} + 2\mu\phi_2 \mathbf{D}_2, & \mathbf{i}_2 &= \mu\phi_2^2 \mathbf{K}_2^{-1} \mathbf{v}_2 \end{aligned} \quad (51)$$

The balance of linear momentum for the two pore-networks becomes

$$\mu\phi_1^2 \mathbf{K}_1^{-1} \mathbf{v}_1 + \text{grad}[\phi_1 p_1] - \text{div}[2\mu\phi_1 \mathbf{D}_1] = \rho_1 \mathbf{b}(\mathbf{x}) \quad (52a)$$

$$\mu\phi_2^2 \mathbf{K}_2^{-1} \mathbf{v}_2 + \text{grad}[\phi_2 p_2] - \text{div}[2\mu\phi_2 \mathbf{D}_2] = \rho_2 \mathbf{b}(\mathbf{x}) \quad (52b)$$

The equations for the balance of mass for the two pore-networks and the rate of mass transfer across the pore-networks (i.e., Eqs. (48c), (48d), and (48e)) remain the same. These governing equations provide a consistent generalization of the classical Brinkman model and the double porosity/permeability model.

4 Mathematical Properties

In this section, we shall establish various mathematical properties that are satisfied by the solutions to the double porosity/permeability model. The mathematical proofs to these properties are provided in the supplementary material which is available under “Supplemental Data” tab for this paper on the ASME Digital Collection. These results are of very high theoretical significance. In addition, they can serve as valuable mechanics-based *a posteriori* measures of the accuracy of numerical solutions of the governing equations. The latter aspect is illustrated in a subsequent paper [47]. We now introduce the required mathematical machinery.

The body force is said to be a conservative vector field if there exists a scalar field ψ such that

$$\gamma \mathbf{b}(\mathbf{x}) = -\text{grad}[\psi]$$

We shall assume a pair of vector fields $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2)$ to be *kinematically admissible* if the following conditions are met:

$$\text{div}[\phi_1 \tilde{\mathbf{v}}_1] + \text{div}[\phi_2 \tilde{\mathbf{v}}_2] = 0 \quad (53a)$$

$$\tilde{\mathbf{v}}_1(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = v_{n1}(\mathbf{x}) \quad (53b)$$

$$\tilde{\mathbf{v}}_2(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = v_{n2}(\mathbf{x}) \quad (53c)$$

Note that a kinematically admissible pair need not satisfy the governing equations for the balance of linear momentum for each pore-network (i.e., Eqs. (4a) and (4b)), or the pressure boundary conditions (i.e., Eqs. (4g) and (4h)). Moreover, it is important to note that the kinematically admissible pair need not satisfy the mass balance equations *individually* (i.e., Eqs. (4c) and (4d)). We shall assume $(\mathbf{v}_1(\mathbf{x}), \mathbf{v}_2(\mathbf{x}))$ to be the pair of true velocity fields if they satisfy all the governing equations under the double porosity/permeability model (i.e., Eqs. (4a)–(4h)). For convenience, we shall denote

$$\alpha_1 = \mu\phi_1^2 \mathbf{K}_1^{-1} \quad \text{and} \quad \alpha_2 = \mu\phi_2^2 \mathbf{K}_2^{-1} \quad (54)$$

Recently, it has been shown that the solutions to the classical Darcy equations satisfy a minimum principle with respect to the mechanical dissipation [48]. Herein, we shall extend this result to the double porosity/permeability model.

THEOREM 1 (Minimum dissipation theorem). *Assume that velocity boundary conditions are enforced on the entire boundary (i.e.,*

$\Gamma_1^v = \Gamma_2^v = \partial\Omega$). *Moreover, $\gamma \mathbf{b}(\mathbf{x})$ is assumed to be a conservative vector field. The dissipation functional is defined as follows:*

$$\Phi[\mathbf{v}_1, \mathbf{v}_2] := \sum_{i=1}^2 \left(\int_{\Omega} \alpha_i \mathbf{v}_i \cdot \mathbf{v}_i d\Omega + \frac{1}{2} \int_{\Omega} \frac{\mu}{\beta} \text{div}[\phi_i \mathbf{v}_i] \text{div}[\phi_i \mathbf{v}_i] d\Omega \right) \quad (55)$$

Then, every kinematically admissible pair $(\tilde{\mathbf{v}}_1(\mathbf{x}), \tilde{\mathbf{v}}_2(\mathbf{x}))$ satisfies

$$\Phi[\mathbf{v}_1, \mathbf{v}_2] \leq \Phi[\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2] \quad (56)$$

where $(\mathbf{v}_1(\mathbf{x}), \mathbf{v}_2(\mathbf{x}))$ denotes the pair of true velocity vector fields. To put it differently, the pair of true velocity vector fields admits the minimum total dissipation among all the possible pairs of kinematically admissible vector fields.

It should be noted that the minimum dissipation theorem is not at odds with the maximization of the rate of dissipation hypothesis, which we discussed in Sec. 3.2.1. The minimum dissipation theorem seeks the minimum among the set of kinematically admissible vector fields. On the other hand, the maximization of rate of dissipation hypothesis maximizes the rate of dissipation among the set of all the fields that satisfy the first and second laws of thermodynamics.

THEOREM 2 (Uniqueness). *The solution under the double porosity/permeability model is unique.*

Unlike the minimum dissipation theorem, the uniqueness theorem does not require the velocity boundary conditions to be prescribed on the entire boundary. That is, the uniqueness has been established under the general boundary conditions provided by Eqs. (4e)–(4h).

Next, we prove a reciprocal relation for double porosity/permeability model. Reciprocal relations are popular in several branches of mechanics. For example, Betti’s reciprocal relation is a classical result in elasticity [49]. Its utility to solve a class of seemingly difficult boundary value problems in linear elasticity is well-documented in the literature; for example, see Refs. [49] and [50]. Recently, reciprocal relations have been obtained for Darcy and Darcy–Brinkman equations in Ref. [48]. It should, however, be noted that the kinematically admissible fields in the aforementioned cases (i.e., elasticity, Darcy and Darcy–Brinkman equations) are different from that of the double porosity/permeability model.

THEOREM 3 (Reciprocal relation). *Let $(\mathbf{v}'_1, p'_1, \mathbf{v}'_2, p'_2)$ and $(\mathbf{v}^*_1, p^*_1, \mathbf{v}^*_2, p^*_2)$ be, respectively, the solutions under the prescribed data-sets $(\mathbf{b}', v'_{n1}, p'_{01}, v'_{n2}, p'_{02})$ and $(\mathbf{b}^*, v^*_{n1}, p^*_{01}, v^*_{n2}, p^*_{02})$. The domain, Ω , and the boundaries, $\Gamma_1^v, \Gamma_1^p, \Gamma_2^v,$ and Γ_2^p , are the same for both prescribed data-sets. The pair of solutions and the pair of prescribed data-sets satisfy the following reciprocal relation:*

$$\begin{aligned} & \int_{\Omega} \phi_1(\mathbf{x}) \gamma \mathbf{b}'(\mathbf{x}) \cdot \mathbf{v}'_1(\mathbf{x}) d\Omega - \int_{\Gamma_1^p} \phi_1(\mathbf{x}) p'_{01}(\mathbf{x}) \mathbf{v}'_1(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) d\Gamma \\ & - \int_{\Gamma_1^v} \phi_1(\mathbf{x}) p'_1(\mathbf{x}) v^*_{n1}(\mathbf{x}) d\Gamma + \int_{\Omega} \phi_2(\mathbf{x}) \gamma \mathbf{b}(\mathbf{x}) \cdot \mathbf{v}^*(\mathbf{x}) d\Omega \\ & - \int_{\Gamma_2^p} \phi_2(\mathbf{x}) p'_{02}(\mathbf{x}) \mathbf{v}'_2(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) d\Gamma - \int_{\Gamma_2^v} \phi_2(\mathbf{x}) p'_2(\mathbf{x}) v^*_{n2}(\mathbf{x}) d\Gamma \\ & = \int_{\Omega} \phi_1(\mathbf{x}) \gamma \mathbf{b}^*(\mathbf{x}) \cdot \mathbf{v}'_1(\mathbf{x}) d\Omega - \int_{\Gamma_1^p} \phi_1(\mathbf{x}) p^*_{01}(\mathbf{x}) \mathbf{v}'_1(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) d\Gamma \\ & - \int_{\Gamma_1^v} \phi_1(\mathbf{x}) p^*_1(\mathbf{x}) v'_{n1}(\mathbf{x}) d\Gamma + \int_{\Omega} \phi_2(\mathbf{x}) \gamma \mathbf{b}^*(\mathbf{x}) \cdot \mathbf{v}'_2(\mathbf{x}) d\Omega \\ & - \int_{\Gamma_2^p} \phi_2(\mathbf{x}) p^*_{02}(\mathbf{x}) \mathbf{v}'_2(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) d\Gamma - \int_{\Gamma_2^v} \phi_2(\mathbf{x}) p^*_2(\mathbf{x}) v'_{n2}(\mathbf{x}) d\Gamma \end{aligned} \quad (57)$$

4.1 Maximum Principle. Maximum principle is one of the basic qualitative properties of second-order elliptic partial differential equations. It can be shown that the pressure under Darcy equations satisfies a maximum principle, which is valid even for heterogeneous and anisotropic permeabilities. To wit, assuming that the pressure boundary conditions are prescribed on the entire boundary, Darcy equations can be rewritten as follows:

$$-\operatorname{div}\left[\frac{1}{\mu}\mathbf{K}(\mathbf{x})\operatorname{grad}[p+\psi]\right]=0 \quad \text{in } \Omega \quad (58a)$$

$$p(\mathbf{x})=p_0(\mathbf{x}) \quad \text{on } \partial\Omega \quad (58b)$$

The above boundary value problem is a second-order elliptic partial differential equation with Dirichlet boundary conditions prescribed on the entire boundary. From the theory of partial differential equations [51], the pressure satisfies:

$$\min_{\mathbf{x}\in\bar{\Omega}}[p_0(\mathbf{x})]\leq p(\mathbf{x})\leq \max_{\mathbf{x}\in\bar{\Omega}}[p_0(\mathbf{x})] \quad \forall \mathbf{x}\in\bar{\Omega} \quad (59)$$

That is, the maximum and minimum pressures occur on the boundary.

On the contrary, the macro- and micropressures under the double porosity/permeability model do not individually enjoy such a maximum principle. One can, however, establish a maximum principle for the difference in pressures in the macropore and micropore networks under some restrictions on the nature of permeabilities and boundary conditions.

THEOREM 4 (Maximum principle). *Assume that the permeabilities are isotropic and homogeneous. That is, $\mathbf{K}_1(\mathbf{x})=k_1\mathbf{I}$ and $\mathbf{K}_2(\mathbf{x})=k_2\mathbf{I}$, where \mathbf{I} is the second-order identity tensor. The entire boundary is prescribed with pressure boundary conditions. That is, $\Gamma_1^p=\Gamma_2^p=\partial\Omega$. The domain Ω is bounded, and the boundary is smooth. Then, the pressure difference in the macropore and micropore networks, $p_1(\mathbf{x})-p_2(\mathbf{x})$, everywhere satisfies*

$$\begin{aligned} \min[0, \min_{\mathbf{x}\in\partial\Omega}[p_{01}(\mathbf{x})-p_{02}(\mathbf{x})]] &\leq p_1(\mathbf{x})-p_2(\mathbf{x}) \\ &\leq \max[0, \max_{\mathbf{x}\in\partial\Omega}[p_{01}(\mathbf{x})-p_{02}(\mathbf{x})]] \end{aligned} \quad (60)$$

The maximum principle for the double porosity/permeability model basically implies that the pressure difference in the micropore and macropore networks everywhere in the domain lies between the corresponding non-negative maximum and the non-positive minimum values on the boundary on which pressures are prescribed.

The main differences between the maximum principles of Darcy equations and the double porosity/permeability model can be summarized as follows:

- (i) The maximum principle for the double porosity/permeability model holds for isotropic and homogeneous permeabilities. There are no such restrictions for Darcy equations.
- (ii) The body force is assumed to be conservative under the maximum principle for Darcy equations. Such a restriction is not needed for the maximum principle for the double porosity/permeability model.
- (iii) The maximum principle for Darcy equations is in terms of the pressure. On the other hand, the maximum principle for the double porosity/permeability model is with respect to the difference in pressures in the macropore and micropore networks.
- (iv) In the case of Darcy equations, the maximum and minimum occur on the boundary. In the case of double porosity/permeability model, the non-negative maximum and the non-positive minimum occur on the boundary.

4.2 Recovery of the Classical Darcy Equations. The solutions (i.e., the pressure and velocity profiles) under the double porosity/permeability model are, in general, more complicated, and qualitatively and quantitatively different from the corresponding ones under the classical Darcy equations. However, there are three scenarios under which the solutions under the double porosity/permeability model can be described using the Darcy equations. That is, we need to show that there is no mass transfer across the two pore-networks under these scenarios. We now discuss these three scenarios, of which two are trivial.

The *first* scenario is when $\phi_2(\mathbf{x})=0$. Physically, this scenario corresponds to the case where there is no micropore network in the porous medium. To see mathematically that Eqs. (4a)–(4d) reduce to the classical Darcy equations, one can appeal to Eq. (4d) and conclude that there is no mass transfer across the pore-networks (i.e., $\chi(\mathbf{x})=0$) in the entire domain. Under this condition, equations for the macropore network (i.e., Eqs. (4a) and (4c)) will reduce to the classical Darcy equations.

The *second* scenario is when $\mathbf{K}_2(\mathbf{x})=\mathbf{0}$. Physically, this scenario corresponds to the case in which the micropores are not interconnected. To show mathematically that one recovers the classical Darcy equations under $\mathbf{K}_2(\mathbf{x})=\mathbf{0}$, one can start with Eq. (6b) and conclude that $\mathbf{u}_2=0$. Equation (6d) will then imply that $\chi(\mathbf{x})=0$ in the entire domain. Similar to the first scenario, the governing equations for the macropore network will reduce to the Darcy equations.

The *third* scenario pertains to the case wherein $\mathbf{K}_1(\mathbf{x})=\mathbf{K}_2(\mathbf{x})$, and the boundary conditions for the macropore and micropore networks are the same. That is, $\Gamma_1^p=\Gamma_2^p$, $\Gamma_1^v=\Gamma_2^v$, $p_{01}(\mathbf{x})=p_{02}(\mathbf{x})$, and $v_{n1}(\mathbf{x})=v_{n2}(\mathbf{x})$. Note that it is *not* necessary for $\phi_1(\mathbf{x})$ to be equal to $\phi_2(\mathbf{x})$. Under these conditions, flow in the porous medium can be modeled using the classical Darcy equations with porosity equal to $\phi(\mathbf{x})=\phi_1(\mathbf{x})+\phi_2(\mathbf{x})$. To see that the mass transfer across the pore networks is zero, one can proceed as follows. For convenience, assume $\mathbf{K}_1(\mathbf{x})=\mathbf{K}_2(\mathbf{x})=\mathbf{K}(\mathbf{x})$. Then, the aforementioned conditions give rise to the following boundary value problem:

$$\mu\mathbf{K}^{-1}(\mathbf{u}_1-\mathbf{u}_2)+\operatorname{grad}[p_1-p_2]=0 \quad \text{in } \Omega \quad (61a)$$

$$\operatorname{div}[\mathbf{u}_1-\mathbf{u}_2]=-\frac{2\beta}{\mu}(p_1-p_2) \quad \text{in } \Omega \quad (61b)$$

$$(\mathbf{u}_1(\mathbf{x})-\mathbf{u}_2(\mathbf{x}))\cdot\hat{\mathbf{n}}(\mathbf{x})=0 \quad \text{on } \Gamma^v=\Gamma_1^v=\Gamma_2^v \quad (61c)$$

$$p_1(\mathbf{x})-p_2(\mathbf{x})=0 \quad \text{on } \Gamma^p=\Gamma_1^p=\Gamma_2^p \quad (61d)$$

Clearly, the pair $p_1(\mathbf{x})-p_2(\mathbf{x})=0$ and $\mathbf{u}_1(\mathbf{x})-\mathbf{u}_2(\mathbf{x})=\mathbf{0}$ is a solution to the above boundary value problem (61a)–(61d). By the uniqueness theorem 2, this is the only solution to the above boundary value problem. Since $p_1(\mathbf{x})=p_2(\mathbf{x})$, the mass transfer across the pore-networks is zero.

In all the above three scenarios, it is important to note that there will be no contribution to the dissipation from the connectors (i.e., conduits/fissures), as there is no flow in the connectors.

5 Analytical Solution Based on Green's Function Approach

In this section, we present an analytical solution procedure for a general boundary value problem arising from the double porosity/permeability model. We provide a formal mathematical derivation based on the Green's function approach.

We start by rewriting the governing Eqs. (6a)–(6h). By eliminating $\mathbf{u}_i(\mathbf{x})$ from these equations, we obtain the following boundary value problem for the macropore network:

$$\operatorname{div}\left[\frac{1}{\mu}\mathbf{K}_1(\mathbf{x})(\gamma\mathbf{b}(\mathbf{x})-\operatorname{grad}[p_1])\right]=\chi(\mathbf{x}) \quad \text{in } \Omega \quad (62a)$$

$$\frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{K}_1(\mathbf{x})(\gamma \mathbf{b}(\mathbf{x}) - \text{grad}[p_1]) = u_{n1}(\mathbf{x}) \quad \text{on } \Gamma_1^v \quad (62b)$$

$$p_1(\mathbf{x}) = p_{01}(\mathbf{x}) \quad \text{on } \Gamma_1^p \quad (62c)$$

By multiplying Eq. (62a) with G_1 , integrating over the domain, employing the Green's identity, and noting the boundary conditions (i.e., Eqs. (62b) and (62c)), we obtain

$$\begin{aligned} & - \int_{\Omega} \text{div} \left[\frac{1}{\mu} \mathbf{K}_1 \text{grad}[G_1] \right] p_1 d\Omega \\ & + \int_{\Gamma_1^p} \frac{1}{\mu} G_1 \hat{\mathbf{n}} \cdot \mathbf{K}_1 (\gamma \mathbf{b} - \text{grad}[p_1]) d\Gamma \\ & + \int_{\Gamma_1} \frac{1}{\mu} \hat{\mathbf{n}} \cdot \mathbf{K}_1 \text{grad}[G_1] p_1 d\Gamma = \int_{\Omega} G_1 \chi d\Omega \quad (63) \\ & + \int_{\Omega} \frac{1}{\mu} \text{grad}[G_1] \cdot \mathbf{K}_1 \gamma \mathbf{b} d\Omega \\ & - \int_{\Gamma_1^v} G_1 u_{n1} d\Gamma - \int_{\Gamma_1^p} \frac{1}{\mu} \hat{\mathbf{n}} \cdot \mathbf{K}_1 \text{grad}[G_1] p_{01} d\Gamma \end{aligned}$$

This suggests to construct the Green's function $G_1(\mathbf{x}, \mathbf{y})$ to be the solution of the following boundary value problem:

$$-\text{div} \left[\frac{1}{\mu} \mathbf{K}_1(\mathbf{x}) \text{grad}[G_1(\mathbf{x}, \mathbf{y})] \right] = \delta(\mathbf{x} - \mathbf{y}) \quad \text{in } \Omega \quad (64a)$$

$$-\frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{K}_1(\mathbf{x}) \text{grad}[G_1(\mathbf{x}, \mathbf{y})] = 0 \quad \text{on } \Gamma_1^v \quad (64b)$$

$$G_1(\mathbf{x}, \mathbf{y}) = 0 \quad \text{on } \Gamma_1^p \quad (64c)$$

where $\delta(\mathbf{x} - \mathbf{y})$ denotes the Dirac-delta distribution [52]. Then the macro-pressure $p_1(\mathbf{x})$ can be written in terms of mass transfer between the pore-networks $\chi(\mathbf{x})$ as follows:

$$\begin{aligned} p_1(\mathbf{x}) &= \int_{\Omega} G_1(\mathbf{x}, \mathbf{y}) \chi(\mathbf{y}) d\Omega_y \\ &+ \int_{\Omega} \frac{1}{\mu} \text{grad}_y [G_1(\mathbf{x}, \mathbf{y})] \cdot \mathbf{K}_1(\mathbf{y}) \gamma \mathbf{b}(\mathbf{y}) d\Omega_y \quad (65) \\ &- \int_{\Gamma_1^v} G_1(\mathbf{x}, \mathbf{y}) u_{n1}(\mathbf{y}) d\Gamma_y \\ &- \int_{\Gamma_1^p} \frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{y}) \cdot \mathbf{K}_1(\mathbf{y}) \text{grad}_y [G_1(\mathbf{x}, \mathbf{y})] p_{01}(\mathbf{y}) d\Gamma_y \end{aligned}$$

where $d\Omega_y$ and $d\Gamma_y$, respectively, denote the volume element and the surface area element with respect to \mathbf{y} -coordinates, and the gradient with respect to \mathbf{y} -coordinates is denoted by $\text{grad}_y[\cdot]$.

By carrying out a similar procedure for the micropore network, the Green's function $G_2(\mathbf{x}, \mathbf{y})$ is taken to be the solution of the following boundary value problem:

$$-\text{div} \left[\frac{1}{\mu} \mathbf{K}_2(\mathbf{x}) \text{grad}[G_2(\mathbf{x}, \mathbf{y})] \right] = \delta(\mathbf{x} - \mathbf{y}) \quad \text{in } \Omega \quad (66a)$$

$$-\frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{K}_2(\mathbf{x}) \text{grad}[G_2(\mathbf{x}, \mathbf{y})] = 0 \quad \text{on } \Gamma_2^v \quad (66b)$$

$$G_2(\mathbf{x}, \mathbf{y}) = 0 \quad \text{on } \Gamma_2^p$$

The micropressure $p_2(\mathbf{x})$ can then be written in terms of $\chi(\mathbf{x})$ as follows:

$$\begin{aligned} p_2(\mathbf{x}) &= - \int_{\Omega} G_2(\mathbf{x}, \mathbf{y}) \chi(\mathbf{y}) d\Omega_y \\ &+ \int_{\Omega} \frac{1}{\mu} \text{grad}_y [G_2(\mathbf{x}, \mathbf{y})] \cdot \mathbf{K}_2(\mathbf{y}) \gamma \mathbf{b}(\mathbf{y}) d\Omega_y \quad (67) \\ &- \int_{\Gamma_2^v} G_2(\mathbf{x}, \mathbf{y}) u_{n2}(\mathbf{y}) d\Gamma_y \\ &- \int_{\Gamma_2^p} \frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{y}) \cdot \mathbf{K}_2(\mathbf{y}) \text{grad}_y [G_2(\mathbf{x}, \mathbf{y})] p_{02}(\mathbf{y}) d\Gamma_y \end{aligned}$$

Note that $G_1(\mathbf{x}, \mathbf{y})$ and $G_2(\mathbf{x}, \mathbf{y})$ are Green's functions for scalar diffusion equations. Since the permeabilities, $\mathbf{K}_1(\mathbf{x})$ and $\mathbf{K}_2(\mathbf{x})$, are symmetric tensors, it is easy to establish that the Green's functions, $G_1(\mathbf{x}, \mathbf{y})$ and $G_2(\mathbf{x}, \mathbf{y})$, are symmetric. That is,

$$G_1(\mathbf{x}, \mathbf{y}) = G_1(\mathbf{y}, \mathbf{x}) \quad \text{and} \quad G_2(\mathbf{x}, \mathbf{y}) = G_2(\mathbf{y}, \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \quad (68)$$

Equations (65) and (67) give rise to the following integral equation for the mass transfer between the pore-networks:

$$\frac{\mu}{\beta} \chi(\mathbf{x}) + \int_{\Omega} (G_1(\mathbf{x}, \mathbf{y}) + G_2(\mathbf{x}, \mathbf{y})) \chi(\mathbf{y}) d\Omega_y = h(\mathbf{x}) \quad (69)$$

where

$$\begin{aligned} h(\mathbf{x}) &:= \int_{\Omega} \frac{1}{\mu} (\mathbf{K}_2(\mathbf{y}) \text{grad}_y [G_2(\mathbf{x}, \mathbf{y})] \\ &- \mathbf{K}_1(\mathbf{y}) \text{grad}_y [G_1(\mathbf{x}, \mathbf{y})]) \cdot \gamma \mathbf{b}(\mathbf{y}) d\Omega_y \\ &+ \int_{\Gamma_1^v} G_1(\mathbf{x}, \mathbf{y}) u_{n1}(\mathbf{y}) d\Gamma_y \\ &+ \int_{\Gamma_1^p} \frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{y}) \cdot \mathbf{K}_1(\mathbf{y}) \text{grad}_y [G_1(\mathbf{x}, \mathbf{y})] p_{01}(\mathbf{y}) d\Gamma_y \quad (70) \\ &- \int_{\Gamma_2^v} G_2(\mathbf{x}, \mathbf{y}) u_{n2}(\mathbf{y}) d\Gamma_y \\ &- \int_{\Gamma_2^p} \frac{1}{\mu} \hat{\mathbf{n}}(\mathbf{y}) \cdot \mathbf{K}_2(\mathbf{y}) \text{grad}_y [G_2(\mathbf{x}, \mathbf{y})] p_{02}(\mathbf{y}) d\Gamma_y \end{aligned}$$

Equation (69) is a nonhomogeneous Fredholm integral equation of second type with symmetric kernel [53]. The symmetry of the kernel stems from the fact that the Green's functions, $G_1(\mathbf{x}, \mathbf{y})$ and $G_2(\mathbf{x}, \mathbf{y})$, are symmetric.

The overall analytical solution procedure can be compactly written as follows:

- (i) Construct the Green's functions, $G_1(\mathbf{x}, \mathbf{y})$ and $G_2(\mathbf{x}, \mathbf{y})$, that are, respectively, the solutions of the boundary value problems given by Eqs. (64) and (66).
- (ii) Using $G_1(\mathbf{x}, \mathbf{y})$ and $G_2(\mathbf{x}, \mathbf{y})$, solve the integral Eq. (69) to obtain the mass transfer between the pore-networks $\chi(\mathbf{x})$.
- (iii) Using the solution for $\chi(\mathbf{x})$, compute the pressures, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$, using Eqs. (65) and (67), respectively.
- (iv) Once the pressures, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$, are known, the discharge velocities, $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$, can be computed using Eqs. (6a) and (6b).

The solution procedure presented above is quite general, as it can be applied even to those problems with anisotropic and heterogeneous medium properties. The procedure is built upon obtaining Green's functions for scalar diffusion equations and solving a linear scalar Fredholm integral equation of second type. There are numerous existing works that provide Green's functions for scalar diffusion equations (e.g., see Ref. [54]). A good deal of work exists on Fredholm integral equations in terms of mathematical theory, analytical solutions, and numerical techniques. For instance, see Refs. [53], [55], and [56].

The boundary value problems corresponding to the Green's functions assumed Γ_1^p and Γ_2^p to be nonempty. That is, it is assumed that there is a (nonempty) portion of the boundary on which Dirichlet boundary conditions are prescribed. One needs to modify the procedure if velocity boundary conditions are prescribed on the entire boundary for a pore-network, which gives rise to a boundary value problem with Neumann boundary conditions for the construction of the Green's function for that particular pore-network. However, one can find in the literature procedures to construct modified Green's functions for diffusion-type equations with Neumann boundary conditions (e.g., see Ref. [54]). This aspect will be illustrated in the next section.

6 Canonical Boundary Value Problems

We now present various boundary value problems to highlight the differences between the bulk response (e.g., single pore-network modeled using Darcy equations) and the one obtained by incorporating the double porosity/permeability model. These problems are specifically designed to be simple, as the primary aim is to illustrate that a number of features and characteristics will be lost in the bulk response. We believe that these findings will be valuable to the subsurface modeling community.

6.1 One-Dimensional Problem #1. Consider a one-dimensional domain of length L . For the macropore network, pressures p_1^L and p_1^R are prescribed on the left and right ends of the domain, respectively. Similarly, pressures of p_2^L and p_2^R are, respectively, prescribed on the left and right ends of the domain for the micro-pore network. The purpose of this boundary value problem is fourfold:

- (i) The problem will be used to illustrate the various steps in the analytical solution procedure that was presented in Sec. 5.
- (ii) It will be shown that the integral Eq. (69) can provide the appropriate and consistent boundary conditions for the mass transfer across the pore-networks in terms of the prescribed velocity and pressure boundary conditions.
- (iii) It will be shown that the maximum and minimum pressures need not occur on the boundary under the double porosity/permeability model for a boundary value problem with pressures prescribed on the *entire* boundary. On the contrary, the maximum and minimum pressures occur on the boundary under Darcy equations for a pressure-prescribed boundary value problem.
- (iv) The maximum principle proposed in Theorem 4 will be verified for this problem.

6.1.1 Nondimensionalization. We take the length of the domain L (L), $p_1^L - p_1^R$ ($ML^{-1}T^{-2}$), and β/μ ($M^{-1}LT$) as the reference quantities. The time and the mass scales are taken as $((\mu/\beta)/(p_1^L - p_1^R))$ (T) and $((L(\mu/\beta)^2)/(p_1^L - p_1^R))$ (M), respectively. We also take the datum for the pressure to be p_1^R . These reference quantities give rise to the following nondimensional quantities, which are denoted by a superposed bar

$$\bar{x} = \frac{x}{L}, \bar{p}_1 = \frac{p_1 - p_1^R}{p_1^L - p_1^R}, \bar{p}_2 = \frac{p_2 - p_1^R}{p_1^L - p_1^R}, \bar{k}_1 = \frac{k_1}{L^2} \quad (71)$$

$$\bar{k}_2 = \frac{k_2}{L^2}, \bar{\mu} = \frac{\mu}{\mu_{\text{ref}}}, \bar{u}_1 = \frac{u_1}{u_{\text{ref}}}, \bar{u}_2 = \frac{u_2}{u_{\text{ref}}}$$

where

$$\mu_{\text{ref}} := \frac{\mu}{\beta}, u_{\text{ref}} := \frac{\beta L}{\mu} (p_1^L - p_1^R) \quad (72)$$

The nondimensional form of the governing equations can be written as follows:

$$\frac{\bar{\mu}}{\bar{k}_1} \bar{u}_1 + \frac{d\bar{p}_1}{d\bar{x}} = 0, \frac{d\bar{u}_1}{d\bar{x}} = -(\bar{p}_1 - \bar{p}_2) \quad \text{in } (0, 1) \quad (73a)$$

$$\frac{\bar{\mu}}{\bar{k}_2} \bar{u}_2 + \frac{d\bar{p}_2}{d\bar{x}} = 0, \frac{d\bar{u}_2}{d\bar{x}} = +(\bar{p}_1 - \bar{p}_2) \quad \text{in } (0, 1) \quad (73b)$$

$$\bar{p}_1(\bar{x} = 0) = 1, \bar{p}_1(\bar{x} = 1) = 0 \quad (73c)$$

$$\bar{p}_2(\bar{x} = 0) = \bar{p}_2^L, \bar{p}_2(\bar{x} = 1) = \bar{p}_2^R \quad (73d)$$

The mass transfer across the two pore-networks takes the following form:

$$\bar{\chi}(\bar{x}) = \bar{p}_2(\bar{x}) - \bar{p}_1(\bar{x}) \quad (74)$$

In this boundary value problem, \bar{k}_1 and \bar{k}_2 are assumed to be independent of \bar{x} . For simplicity, we drop the over-lines, as all the quantities below will be nondimensional.

6.1.2 Analytical Solution. For convenience, let us introduce the following parameter:

$$\eta := \sqrt{\frac{\mu(k_1 + k_2)}{k_1 k_2}} \quad (75)$$

Note that η is inversely proportional to the square root of the harmonic average of the permeabilities in the macropore and micropore networks. The Green's functions $G_1(x, y)$ and $G_2(x, y)$ will be

$$\frac{k_1}{\mu} G_1(x, y) = \frac{k_2}{\mu} G_2(x, y) = \begin{cases} x - xy & x \leq y \\ y - xy & x > y \end{cases} \quad (76)$$

The integral equation for the mass transfer becomes

$$\chi(x) - \int_0^1 \eta^2 xy \chi(y) dy + \int_0^x \eta^2 y \chi(y) dy + \int_x^1 \eta^2 x \chi(y) dy = h(x) \quad (77)$$

Using the Leibniz integral rule and noting that $h''(x) = 0$, the above equation implies that

$$\frac{d^2 \chi}{dx^2} = \eta^2 \chi \quad (78)$$

The boundary conditions for the mass transfer $\chi(x)$ in terms of the prescribed pressure boundary conditions take the following form:

$$\chi(x = 0) = h(x = 0^+) = p_2^L - 1 \quad (79a)$$

$$\chi(x = 1) = h(x = 1^-) = p_2^R \quad (79b)$$

The solution of the boundary value problem given by Eqs. (78) and (79a)–(79b), which will also be the solution of the integral Eq. (77), takes the following form:

$$\chi(x) = C_1 \exp[\eta x] + C_2 \exp[-\eta x] \quad (80)$$

where

$$C_1 = \frac{p_2^R + (1 - p_2^L) \exp[-\eta]}{\exp[\eta] - \exp[-\eta]} \quad \text{and} \quad (81)$$

$$C_2 = -\frac{p_2^L + (1 - p_2^R) \exp[\eta]}{\exp[\eta] - \exp[-\eta]}$$

The solution for the pressures in the macropore and micropore networks can be written as follows:

$$p_1(x) = \underbrace{1-x}_{\text{Darcy solution}} - \underbrace{\frac{\mu}{k_1 \eta^2} \left\{ (1-p_2^L)(1-x) - p_2^R x + C_1 \exp[\eta x] + C_2 \exp[-\eta x] \right\}}_{\text{deviation due to mass transfer}} \quad (82a)$$

$$p_2(x) = \underbrace{p_2^L(1-x) + p_2^R x}_{\text{Darcy solution}} + \underbrace{\frac{\mu}{k_2 \eta^2} \left\{ (1-p_2^L)(1-x) - p_2^R x + C_1 \exp[\eta x] + C_2 \exp[-\eta x] \right\}}_{\text{deviation due to mass transfer}} \quad (82b)$$

The solution for the discharge velocities in the pore-networks can be written as follows:

$$u_1(x) = \underbrace{\frac{k_1}{\mu}}_{\text{Darcy solution}} + \underbrace{\frac{1}{\eta^2} (p_2^L - p_2^R - 1) + \frac{1}{\eta} (C_1 \exp[\eta x] - C_2 \exp[-\eta x])}_{\text{deviation due to mass transfer}} \quad (83a)$$

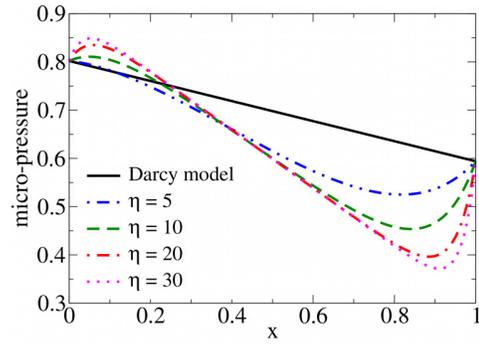
$$u_2(x) = \underbrace{\frac{k_2}{\mu} (p_2^L - p_2^R)}_{\text{Darcy solution}} - \underbrace{\frac{1}{\eta^2} (p_2^L - p_2^R - 1) - \frac{1}{\eta} (C_1 \exp[\eta x] - C_2 \exp[-\eta x])}_{\text{deviation due to mass transfer}} \quad (83b)$$

The deviation of the solution under the double porosity/permeability model from the corresponding one under Darcy equations is indicated in the above expressions for the analytical solution. One fact that is clear from the analytical solution is that the nature of the solution (i.e., the pressure and velocity profiles) depends on the parameter η .

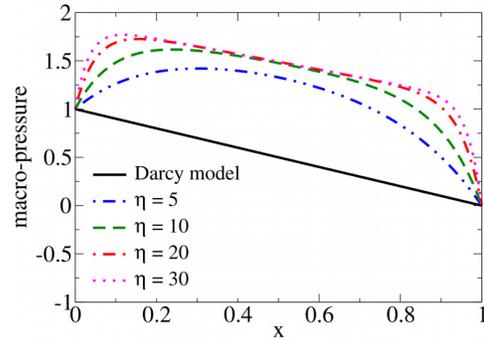
Figure 3 illustrates that the maximum and minimum pressures for macropore and micropore networks need not occur on the boundary. On the other hand, as mentioned earlier, the maximum and minimum pressures occur on the boundary under Darcy equations for a boundary value problem with pressures prescribed on the entire boundary. Figure 4 numerically verifies the maximum principle proposed in Theorem 4 for the double porosity/permeability model. As one can see from this figure, the non-negative maximum and the nonpositive minimum of the pressure difference, $p_1(x) - p_2(x)$, occur on the boundary under the double porosity/permeability model.

6.2 One-Dimensional Problem #2. In this problem, pressure boundary conditions are applied to the macropore network, and no-flux (i.e., zero normal velocity) boundary conditions are enforced on the micropore network. This problem highlights the following important points:

- (i) One can have discharge (i.e., nonzero velocity) in the micropore network even if the micropore network does not extend to the boundary (i.e., there is no discharge on the boundary of the micropore network). This implies that, for complicated porous media, it is essential to know the internal pore-structure (e.g., using μ -CT [20]). It is not sufficient to know the surface pore-structure on the boundary.
- (ii) One can find the solution uniquely for all the fields (i.e., pressures, velocities, and mass transfer) even if the



(a) $p_2^R < p_2^L < 1$



(b) $1 < p_2^R < p_2^L$

Fig. 3 One-dimensional problem #1: variation of micropressure and macropressure in the one-dimensional (1D) domain. For comparison, the analytical solution under Darcy equations is also plotted. The maximum and minimum pressures in the pore-networks need not occur on the boundary in the case of double porosity/permeability model. The parameter η is defined in equation (75): (a) $p_2^R < p_2^L < 1$ and (b) $1 < p_2^R < p_2^L$.

velocity boundary conditions are prescribed on the entire boundary for one of the pore-networks. On the other hand, one cannot find the pressure uniquely under Darcy equations if the velocity boundary conditions are prescribed on the entire boundary.

- (iii) This problem will be utilized to illustrate the construction of modified Green's function for problems involving velocity boundary conditions for either macropore or micropore network.

We shall employ the same reference quantities, as defined in problem #1. The nondimensional form of the governing equations for this boundary value problem can be written as follows:

$$\frac{\mu}{k_1} u_1 + \frac{dp_1}{dx} = 0, \quad \frac{du_1}{dx} = -(p_1 - p_2) \quad \text{in } (0, 1) \quad (84a)$$

$$\frac{\mu}{k_2} u_2 + \frac{dp_2}{dx} = 0, \quad \frac{du_2}{dx} = +(p_1 - p_2) \quad \text{in } (0, 1) \quad (84b)$$

$$p_1(x=0) = 1, \quad p_1(x=1) = 0 \quad (84c)$$

$$u_2(x=0) = 0, \quad u_2(x=1) = 0 \quad (84d)$$

The expression for the mass transfer across the pore-networks is the same as before.

6.2.1 Analytical Solution. The Green's function $G_1(x, y)$ will be

$$G_1(x, y) = \frac{\mu}{k_1} \begin{cases} x - xy & x \leq y \\ y - xy & x > y \end{cases} \quad (85)$$

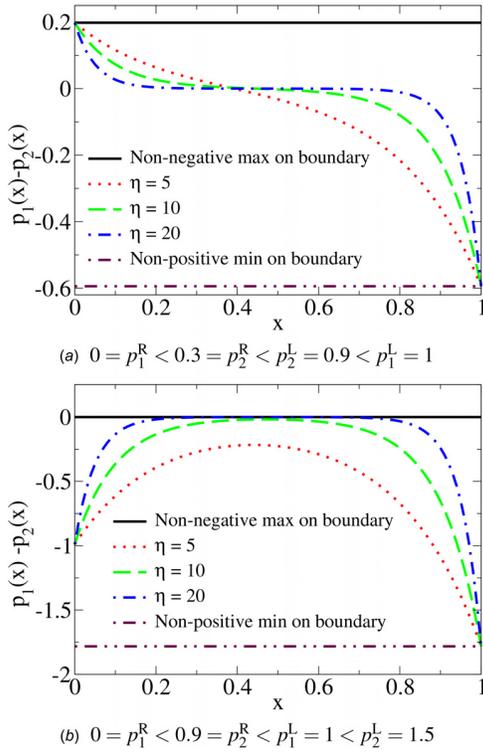


Fig. 4 One-dimensional problem #1: This figure numerically verifies the maximum principle given by Theorem 4. According to the maximum principle, $p_1(x) - p_2(x)$ in the entire domain lies between the non-negative maximum and nonpositive minimum values on the boundary. Note that the medium properties are isotropic and homogeneous: (a) $0 = p_1^R < 0.3 = p_2^R < p_2^L = 0.9 < p_1^L = 1$ and (b) $0 = p_1^R < 0.9 = p_2^R < p_1^L = 1 < p_2^L = 1.5$.

Since the boundary value problem for the micropore network has Neumann boundary conditions on the entire boundary, one needs to modify the procedure for finding the Green's function. Following the technique provided in Ref. [54], the Green's function $G_2(x, y)$ is constructed in such a way that it satisfies the following boundary value problem

$$\begin{aligned}
 -\frac{\mu}{k_2} \frac{d^2 G_2}{dx^2} &= \delta(x - y) - 1 \quad \text{in } (0, 1) \\
 \frac{\mu}{k_2} \frac{dG_2}{dx} \Big|_{x=0} &= 0 \\
 \frac{\mu}{k_2} \frac{dG_2}{dx} \Big|_{x=1} &= 0
 \end{aligned} \tag{86}$$

The Green's function $G_2(x, y)$ takes the following form:

$$G_2(x, y) = \frac{\mu}{k_2} \begin{cases} \frac{x^2}{2} - y + C_0 & \text{for } x \leq y \\ \frac{x^2}{2} - x + C_0 & \text{for } x > y \end{cases} \tag{87}$$

where C_0 is a constant which needs to be determined. It should be noted that C_0 is independent of x , but could depend on y . The integral equation for the mass transfer becomes

$$\begin{aligned}
 \chi(x) + \int_0^1 p_2(y) dy - \frac{1}{k_1} \int_0^1 xy \chi(y) dy + \frac{1}{k_2} \int_0^1 (C_0 + x^2/2) \chi(y) dy \\
 + \int_0^x \left(\frac{y}{k_1} - \frac{x}{k_2} \right) \chi(y) dy + \int_x^1 \left(\frac{x}{k_1} - \frac{y}{k_2} \right) \chi(y) dy = h(x)
 \end{aligned} \tag{88}$$

Using the Leibniz integration rule, the integral Eq. (88) can be shown to be equivalent to the following differential equation:

$$\frac{d^2 \chi}{dx^2} + \frac{1}{k_2} \int_0^1 \chi(y) dy = \eta^2 \chi(x) \tag{89}$$

where the parameter η is defined in Eq. (75). The solution for the above differential equation takes the following form:

$$\chi(x) = D_1 \exp[\eta x] + D_2 \exp[-\eta x] \tag{90}$$

The boundary conditions give rise to

$$\begin{aligned}
 D_1 &= \frac{k_1 + k_2}{k_1 \eta (\exp[\eta] + 1) + 2k_2 (\exp[\eta] - 1)} \\
 D_2 &= -\exp[\eta] D_1
 \end{aligned} \tag{91}$$

The analytical solution can be compactly written as follows:

$$\begin{aligned}
 p_1(x) &= 1 - x - \frac{1 - 2x}{2 + \coth[\eta/2] \frac{k_1 \eta}{k_2}} \\
 &\quad - \frac{\mu}{k_1} \frac{1}{\eta^2} (D_1 \exp[\eta x] + D_2 \exp[-\eta x])
 \end{aligned} \tag{92a}$$

$$\begin{aligned}
 p_2(x) &= 1 - x - \frac{1 - 2x}{2 + \coth[\eta/2] \frac{k_1 \eta}{k_2}} \\
 &\quad + \frac{\mu}{k_2} \frac{1}{\eta^2} (D_1 \exp[\eta x] + D_2 \exp[-\eta x])
 \end{aligned} \tag{92b}$$

$$\begin{aligned}
 u_1(x) &= \frac{k_1}{\mu} - \frac{k_1}{\mu} \frac{2}{2 + \coth[\eta/2] \frac{k_1 \eta}{k_2}} \\
 &\quad + \frac{1}{\eta} (D_1 \exp[\eta x] - D_2 \exp[-\eta x])
 \end{aligned} \tag{92c}$$

$$\begin{aligned}
 u_2(x) &= \frac{k_2}{\mu} - \frac{k_2}{\mu} \frac{2}{2 + \coth[\eta/2] \frac{k_1 \eta}{k_2}} \\
 &\quad - \frac{1}{\eta} (D_1 \exp[\eta x] - D_2 \exp[-\eta x])
 \end{aligned} \tag{92d}$$

Note that one cannot find $p_2(x)$ uniquely under Darcy equations, as the boundary conditions for the micropore network are all velocity boundary conditions. All one can say about the solution for $p_2(x)$ under Darcy equations is that it is an arbitrary constant. On the other hand, one can find uniquely the solution for $p_2(x)$ under the double porosity/permeability model.

6.2.2 Comparison With Darcy Equations. If only the macropore network is present, the boundary conditions of the macropore network imply that the pressure and the velocity take the following forms:

$$u_1(x) = \frac{k_1}{\mu} \quad \text{and} \quad p_1(x) = 1 - x \tag{93}$$

If only the micropore network is present, the boundary conditions of the micropore network imply that

$$u(x) = 0 \quad \text{and} \quad p(x) = \text{an arbitrary constant} \tag{94}$$

If the standard permeability test is performed on a porous medium (which has both pore-networks), the resulting permeability can be expressed as follows:

$$k_{\text{eff}} = \mu u_1(0) = \frac{k_1 + k_2}{2 \tanh[\eta/2] \frac{k_2}{k_1 \eta} + 1} \quad (95)$$

Remark 6.1. In order to provide an insight into the above-mentioned effective permeability, it is necessary to outline the standard permeability test. A cylindrical porous sample of length L is subjected to a pressure p on one end, and the other end is subjected to the atmospheric pressure, p_{atm} . The lateral side of the porous cylinder is sealed (i.e., there is no fluid discharge from the lateral side). The pressure along the length of the cylinder is assumed to be linear, which means that the flow is assumed to be uniform along the length. By measuring the rate of discharge of the fluid, Q , and calculating the area of the cross section, A , the permeability is given by the following formula:

$$\frac{k}{\mu} \left(\frac{p - p_{\text{atm}}}{L} \right) = \frac{Q}{A} \quad (96)$$

Clearly, $Q/A = u(L)$, which is the velocity of fluid at the end where the rate of discharge is measured. Since, the velocity is uniform, we have

$$Q/A = u(L) = u(0) \quad (97)$$

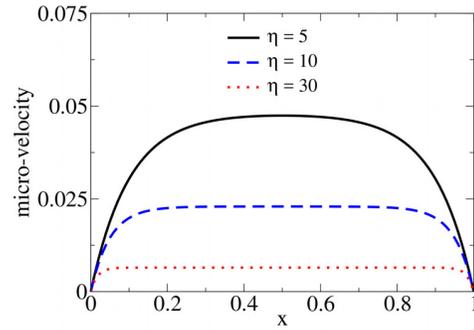
Based on the nondimensionalization employed in this problem (i.e., see Eq. (84) and the discussion preceding this equation), we have $(p - p_{\text{atm}})/L = 1$. Hence, the permeability based on the Darcy model will be $k = \mu u(0)$.

To translate this result to the porous medium with two pore-networks, we note that there is no (net) discharge from the micropore network in this problem (i.e., the velocity in the micropore network is zero at both the ends of the specimen). [However, there could be nonzero velocity in the micropore network within the porous sample.] This implies that the fluid discharge from the porous sample is entirely due to the velocity in the macropore network (i.e., $u(0) = u_1(0)$).

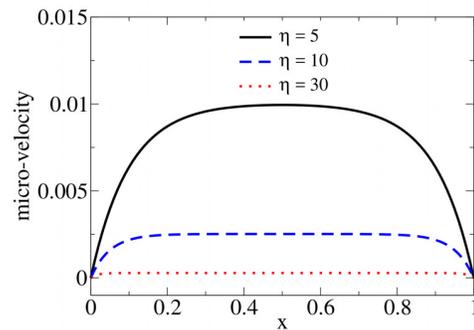
Equation (95) implies that one can relate the experimental value of the effective permeability to the permeabilities of macro- and micropore networks. This clearly shows the need to know the internal pore-structure for an accurate modeling of porous media.

Figure 5 shows the variation of velocity in micropore network and the mass transfer across the pore-networks for various values of η and for two different cases $k_1 < k_2$ and $k_1 > k_2$. It is observed that although there is no supply of fluid on the boundaries of the micropore network, there will still be discharge (i.e., nonzero velocity) within the micropore network. This reveals that the internal pore-structure is an important factor characterizing the flow in a complicated porous medium. One can analyze the internal pore-structure using modern techniques such as μ -CT. In other words, the surface pore-structure cannot solely specify the flow within the domain. Moreover, whether the permeability of the macropores is larger or smaller than the permeability of the micropores, we will still have flow in the micropore network.

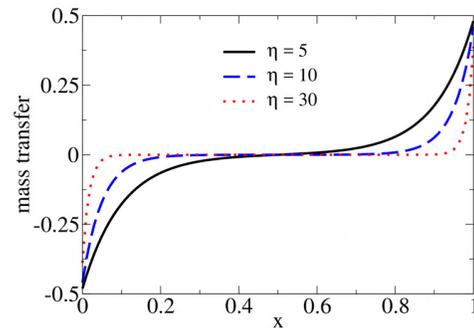
Figure 6 compares the velocities under the double porosity/permeability model and the Darcy equations. Here, the permeability used in the Darcy equations is the effective permeability introduced in Eq. (95). Macro- and microvelocities and their summation (i.e., $u_1 + u_2$) under the double porosity/permeability model as well as the velocity under the Darcy equations are displayed. As can be seen for both cases $k_1 > k_2$ and $k_1 < k_2$, under the Darcy equations the velocity throughout the domain for this one-dimensional boundary value problem is a horizontal line where the constant value is equal to the summation of the macro- and microvelocities under the double porosity/permeability model. This implies that the effective permeability $k = k_{\text{eff}}$, which is obtained by the classical Darcy experiment, cannot completely capture the complex internal pore-structure of the porous medium. This is due to the fact that the experimental value obtained for k_{eff}



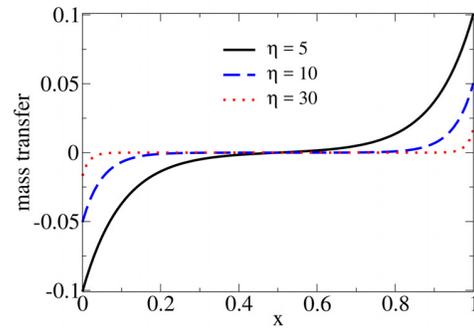
(a) Micro-velocity for $k_1 < k_2$



(b) Micro-velocity for $k_1 > k_2$



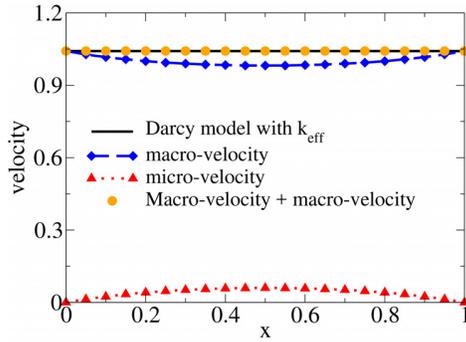
(c) Mass transfer for $k_1 < k_2$



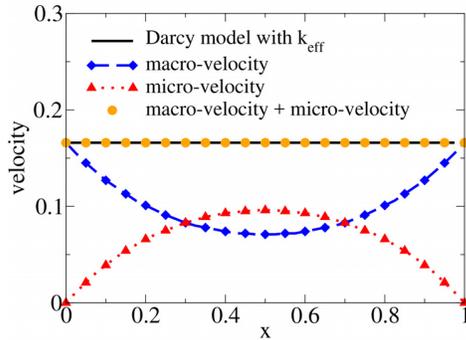
(d) Mass transfer for $k_1 > k_2$

Fig. 5 One-dimensional problem #2: Variation of the micro-velocity and mass transfer for various η values for the cases $k_1 < k_2$ and $k_1 > k_2$. Although there is no supply of fluid on the boundaries of the micropore network, there is still a discharge (i.e., nonzero velocity) in the micropore network, and there is a mass transfer across the pore-networks: (a) microvelocity for $k_1 < k_2$, (b) microvelocity for $k_1 > k_2$, (c) mass transfer for $k_1 < k_2$, and (d) mass transfer for $k_1 > k_2$.

does not account for the case of multiple pore-networks within the domain. It just assumes a single pore-network, and the effective permeability is calculated based on the surface pore-structure of the specimen.



(a) Case 1: $k_1 = 1.0$ and $k_2 = 0.1$



(b) Case 2: $k_1 = 0.1$ and $k_2 = 1.0$

Fig. 6 One-dimensional problem #2: This figure compares the velocities under double porosity/permeability model and Darcy model for the cases $k_1 > k_2$ and $k_1 < k_2$. Macro- and microvelocities and their summation under the double porosity/permeability model as well as the velocity under the Darcy model with $k = k_{\text{eff}}$ are displayed. As can be seen, k_{eff} obtained by the classical Darcy experiment cannot capture the complex internal pore-structure: (a) case 1: $k_1 = 1.0$ and $k_2 = 0.1$ and (b) case 2: $k_1 = 0.1$ and $k_2 = 1.0$.

6.3 Two-Dimensional Boundary Value Problem. This problem pertains to the flow of water in candle filters, which are widely used for purifying drinking water [57]. Consider a circular disk of inner radius $r_i = a$ and outer radius of $r_o = 1$. The inner surface of the cylinder is subjected to a pressure, and the outer surface of the cylinder is exposed to the atmosphere. For the micropore network, there is no discharge from the inner and outer surfaces of the cylinder. Figure 7 provides a pictorial description of the problem.

We shall employ cylindrical polar coordinates. Noting the underlying symmetry in the problem, the variables, u_1 , u_2 , p_1 , and p_2 , are assumed to be functions of r only. The governing equations can be written as follows:

$$\frac{\mu}{k_1} u_1 + \frac{dp_1}{dr} = 0, \quad \frac{1}{r} \frac{d(ru_1)}{dr} + (p_1 - p_2) = 0 \quad \forall r \in (a, 1) \quad (98a)$$

$$\frac{\mu}{k_2} u_2 + \frac{dp_2}{dr} = 0, \quad \frac{1}{r} \frac{d(ru_2)}{dr} - (p_1 - p_2) = 0 \quad \forall r \in (a, 1) \quad (98b)$$

$$p_1(r = a) = 1, \quad p_1(r = 1) = 0 \quad (98c)$$

$$u_2(r = a) = 0, \quad u_2(r = 1) = 0 \quad (98d)$$

This implies that the mass transfer $\chi(r)$ satisfies the following differential equation:

$$\chi'' + \frac{1}{r} \chi' - \eta^2 \chi = 0 \quad (99)$$

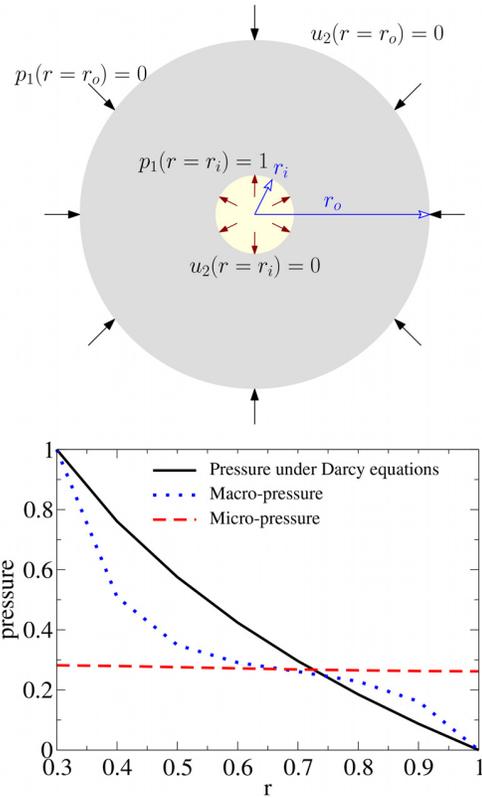


Fig. 7 The top figure provides a pictorial description of the boundary value problem. There is no discharge on the inner and outer surfaces of the micropore network. For the macropore network, the inner surface is subjected to a pressure of unity, and the outer surface is subjected to a pressure of zero. The bottom figure illustrates that the macropressure under the double porosity/permeability model is qualitatively different from the pressure under Darcy equations.

which is a (homogeneous) modified Bessel ordinary differential equation [58]. A general solution to the above ordinary differential equation can be written as follows:

$$\chi(r) = p_2(r) - p_1(r) = C_3 I_0(\eta r) + C_4 K_0(\eta r) \quad (100)$$

where $I_0(z)$ and $K_0(z)$ are, respectively, zeroth-order modified Bessel functions of first and second kinds. Noting that $I'_0(z) = I_1(z)$ and $K'_0(z) = -K_1(z)$, the analytical solution can be written as follows:

$$p_1(r) = \frac{\ln[r]}{\ln[a]} C_1 + C_2 - \frac{\alpha_1}{\eta^2} (C_3 I_0(\eta r) + C_4 K_0(\eta r)) \quad (101a)$$

$$p_2(r) = \frac{\ln[r]}{\ln[a]} C_1 + C_2 + \frac{\alpha_2}{\eta^2} (C_3 I_0(\eta r) + C_4 K_0(\eta r)) \quad (101b)$$

$$u_1(r) = -\frac{k_1}{\mu} \frac{C_1}{r \ln[a]} + \frac{1}{\eta} (C_3 I_1(\eta r) - C_4 K_1(\eta r)) \quad (101c)$$

$$u_2(r) = -\frac{k_2}{\mu} \frac{C_1}{r \ln[a]} - \frac{1}{\eta} (C_3 I_1(\eta r) - C_4 K_1(\eta r)) \quad (101d)$$

The boundary conditions give rise to the following coefficients:

$$C_1 = \delta \eta^2 a \ln[a] (I_1(a\eta) K_1(\eta) - I_1(\eta) K_1(a\eta)) x_2 \quad (102a)$$

$$C_2 = \delta(-1 + a\eta I_1(a\eta)K_0(\eta) + a\eta I_0(\eta)K_1(a\eta))\alpha_1 \quad (102b)$$

$$C_3 = \delta\eta^3(aK_1(a\eta) - K_1(\eta)) \quad (102c)$$

$$C_4 = \delta\eta^3(aI_1(a\eta) - I_1(\eta)) \quad (102d)$$

where

$$\begin{aligned} \delta^{-1} = & \{-2 + a\eta I_1(a\eta)K_0(\eta) + \eta I_1(\eta)K_0(a\eta) \\ & + \eta I_0(a\eta)K_1(\eta) + a\eta I_0(\eta)K_1(a\eta)\}\alpha_1 \\ & + \eta^2 a \ln[a](I_1(a\eta)K_1(\eta) - I_1(\eta)K_1(a\eta))\alpha_2 \end{aligned} \quad (103)$$

For comparison, the pressure and the discharge velocity under Darcy equations with constant permeability k and with boundary conditions $p(r=a) = 1$ and $p(r=1) = 0$ can be written as follows:

$$p(r) = \frac{\ln[r]}{\ln[a]} \quad \text{and} \quad u(r) = -\frac{k}{\mu r \ln[a]} \quad (104)$$

It is evident that the velocity and pressure profiles under the double porosity/permeability model are much more complicated than the corresponding profiles under Darcy equations. Figure 7 illustrates the qualitative difference between the pressures under the double porosity/permeability model and Darcy equations. The graph of the pressures under Darcy equations is always convex, while the graph of the macropressure under the double porosity/permeability model has both convex and concave parts. It should also be noted that, although there is no discharge from the micropore network on the boundary, there is discharge in the micropore network within the domain.

7 Conclusions

In this paper, several contributions have been made to the modeling of incompressible single-phase fluid flow in rigid porous media with double porosity/permeability. *First*, a thermodynamic basis for models studying flow in porous media exhibiting double porosity/permeability has been provided using the maximization of rate of dissipation hypothesis. This model nicely allows for further generalizations of the existing models. The mass transfer across the macropore and micropore networks has been obtained in a systematic manner by treating it as an internal variable and maximizing a prescribed (physical) dissipation functional. The resulting mathematical model gives rise to a system of elliptic partial differential equations in the case of a steady-state response and a system of parabolic partial differential equations in the case of a transient response. *Second*, various mathematical properties that the solutions under the double porosity/permeability model satisfy have been presented along with their proofs. *Third*, a maximum principle has been established for the double porosity/permeability model. The main differences between the maximum principles of Darcy equations and the double porosity/permeability model have been discussed. *Fourth*, an analytical solution procedure based on the Green's function method has been presented for a general boundary value problem under the double porosity/permeability model. *Last but not least*, using the analytical solutions of some canonical problems, the salient features of the pressures and velocities in the macropore and micropore networks under the double porosity/permeability model have been highlighted. Some of the significant findings of the paper can be summarized as follows:

- (C1) In general, the pressure and velocity profiles under the double porosity/permeability model are qualitatively and quantitatively different from the corresponding ones under the classical Darcy equations (e.g., see the maximum principle discussed in Sec. 4.1 and illustrated in Figs. 3 and 4, and the canonical problems outlined in Sec. 4 and the

associated results provided in Figs. 6 and 7). These differences can be attributed to the complex nature of a porous medium that exhibits double porosity/permeability. However, there are situations under which the solutions under the double porosity/permeability model can be adequately described by Darcy equations (see Sec. 4.2).

- (C2) The maximum and minimum pressures need not occur on the boundary under the double porosity/permeability model. This is in contrast with the case of Darcy equations under which the maximum and minimum pressures occur on the boundary for a pressure-prescribed boundary value problem.
- (C3) There will be discharge in the micropore network even if there is no fluid supply on the boundaries of the micropore network. Therefore, it can be concluded that the surface pore-structure is not the only factor in characterizing the flow through a complex porous medium which highlights the need to use modern techniques (e.g., μ -CT) for studying the internal pore-structure.
- (C4) There will be mass transfer across the two pore-networks whether the permeability of the macropore network is greater than the permeability of the micropore network or vice-versa. This means that the path that the fluid takes is not necessarily through the network with higher permeability.

The proposed theoretical framework can be extended to the study of the flow of multiphase fluids in a porous medium with double porosity/permeability. A systematic study comparing the linear mass transfer model and multirate mass transfer models is a worthy scientific endeavor. Another research endeavor can be toward extending the proposed framework to coupling the deformation of the porous solid with the flow in a porous medium with double porosity/permeability.

Nomenclature

Symbols.

- A_i = specific Helmholtz potential
 $\mathbf{b}(\mathbf{x})$ = the specific body force
 $\text{cl}(\cdot)$ = set closure
 \mathbf{D}_i = symmetric part of gradient of velocity
 $(D^{(i)}(\cdot)/Dt)$ = material time derivative
 \mathbf{F}_i = gradient of motion tensor
 \mathbf{i}_i = interaction term
 k_i = scalar permeability
 \mathbf{K}_i = permeability tensor
 \mathbf{L}_i = gradient of velocity tensor
 m_i = rate of mass transfer (due to chemical reactions, phase change, etc.)
 $\hat{\mathbf{n}}(\mathbf{x})$ = unit outward normal to the boundary
 \mathbf{p}_i = material (reference) coordinate
 $p_i(\mathbf{x})$ = pressure scalar field
 p_{0i} = prescribed pressure
 $\mathbf{q}_i(\mathbf{x})$ = heat flux
 r_i = (external) specific heat supply
 \mathbf{T}_i = partial Cauchy stress tensor
 $\mathbf{u}_i(\mathbf{x})$ = discharge (or Darcy) velocity vector field
 u_{ni} = prescribed normal component of Darcy velocity
 U_i = specific internal energy
 $\mathbf{v}_i(\mathbf{x})$ = true (or seepage) velocity vector field
 v_{ni} = prescribed normal component of true velocity
 $\tilde{\mathbf{v}}_i(\mathbf{x})$ = kinematically admissible velocity vector field
 \mathbf{x} = spatial coordinate
 α_i = drag coefficient
 β = dimensionless characteristic of the porous medium
 γ = true density of fluid
 Γ_i^p = pressure boundary
 Γ_i^v = velocity boundary
 \hat{e}_i^s = energy supply

ζ = rate of dissipation per unit volume
 ζ_{MT} = dissipation due to mass transfer
 η_i = specific entropy
 θ_i = temperature
 μ = coefficient of viscosity of fluid
 ρ_i = bulk density
 $\phi_i(\mathbf{x})$ = porosity (volume fractions)
 Φ = dissipation functional
 $\chi(\mathbf{x})$ = mass transfer across the two pore-networks
 Ψ = rate of entropy production per unit volume
 Ω = bounded domain
 $\partial\Omega$ = piecewise smooth boundary

Subscripts

1 = macropore network
 2 = micropore network

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