# Effect of Stress Singularity Magnitude on Scaling of Strength of Quasi-Brittle Structures

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**Abstract:** Engineering structures are often designed to have complex geometries, which could introduce stress singularities that are weaker than the conventional $-1/2$ crack-tip singularity. Extrapolating the results of small-scale laboratory tests to predict the response of a full-scale structure comprised of quasi-brittle materials requires an understanding of how the weak stress singularities modify the classical energetic and statistical scaling theories of quasi-brittle fracture. Through a theoretical and numerical study, a new scaling law for quasi-brittle fracture is derived, which explicitly relates the nominal structural strength to the structure size and the magnitude of the stress singularity. The theoretical analysis is based on a generalized weakest-link model that combines the energetic scaling of fracture with the finite weakest-link model. The model captures the transition from the energetic scaling to statistical scaling as the strength of the stress singularity diminishes. The new scaling law is in close agreement, for the entire range of stress singularities, with the size effect curves predicted through finite-element simulations of concrete beams containing an arbitrary-angle V-notch under Mode-I fracture. **DOI: 10.1061/(ASCE)EM.1943-7889.0000693.** © *2014 American Society of Civil Engineers.*

**Author keywords:** Size effect; Weakest-link model; Quasi-brittle fracture; Stress singularity; Asymptotic analysis.



## Introduction

The designs of large-scale engineering structures, such as bridges, dams, aircraft, and ships, usually rely on the results of reduced-scale laboratory testing. Therefore, understanding the size effect on structural strength is of paramount importance for ensuring a safe and reliable design. Many of these engineering structures are made of quasi-brittle materials, which are brittle heterogeneous in nature, as exemplified by concrete, fiber-reinforced composites, ceramics, rocks, and others at the microscales and nanoscales. It is well known that quasi-brittle structures exhibit a size-dependent transitional failure behavior, where small-size structures fail in a quasi-plastic manner and large-size structures fail in a perfectly brittle manner. The underlying reason is that the size of the material inhomogeneities is not negligible compared with the structure size. The transition from ductile to brittle response can be characterized by the so-called size effect on the nominal structural strength. The nominal strength is a load parameter usually defined as $\sigma_N = cP_{\max}/bD$ for two-dimensional (2D) structures and $\sigma_N = cP_{\max}/D^2$ for three-dimensional (3D) structures, where $P_{\max}$ is the maximum load capacity of the structure, $D$ is the characteristic size of the structure to be scaled, $b$ is the width of the structure in the third (transverse) direction, and $c$ is a constant such that $\sigma_N$ reduces to some familiar parameter such as the maximum stress in the structure in the absence of the stress concentration. So far, two independent mechanisms have been identified that explain the size effect on the nominal strength (Bažant 2004, 2005): one is based on the statistics of random material strength, and the other is based on the energetic argument of material fracture.

The statistical scaling theory usually applies to structures for which the peak load is reached when a macrocrack initiates from one material representative volume element (RVE). In other words, the failure statistics of the structure can be described by the weakest-link model, where each link corresponds to one RVE. Research on statistical scaling has a long and rich history dating back to Leonardo da Vinci (1945, p. 546), who first speculated this type of scaling phenomenon. Mariotte (1686, p. 249) proposed a qualitative explanation of the statistical size effect that attributes the size effect to the randomness of material strength. The formal mathematical framework of the statistical size effect on structural strength initiated with Fisher and Tippet's (1928) seminal work on the extreme value statistics. Weibull (1939) independently investigated the extreme value statistics and applied it to describe the random material strength, i.e., the Weibull distribution. The Weibull distribution directly yields the Weibull size effect, which has been successfully applied to perfectly brittle structures including fine-grain engineering and dental ceramics. The applicability of the Weibull distribution for brittle structures is attributed to the fact that the size of their RVEs is orders of magnitude smaller than the structural dimensions. This implies an infinite weakest-link model (i.e., infinite number of RVEs) and the corresponding probability distribution of structural strength can be described by the extreme value statistics.

In contrast with perfectly brittle structures, the RVE size of quasi-brittle structures is not negligible compared with the structure size, and thus, the number of RVEs in the weakest-link model must be finite. The infinite weakest-link model only requires knowledge of the far-left tail of the cumulative distribution function (cdf) of RVE strength. But the finite weakest-link model requires the entire strength cdf of one RVE to be known. Recent studies (Bažant and Pang 2006, 2007; Bažant et al. 2009; Le et al. 2011) have shown that the strength cdf of one RVE can be derived from atomistic fracture

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mechanics and a hierarchical multiscale transition model. The resulting finite weakest-link model predicts an intricate scale effect on the strength cdf of the structure, varying from the Gaussian distribution grafted by a power-law tail for small-size structures to the Weibull distribution for large-size structures. Such a non-Weibullian strength cdf for intermediate-size structures agrees well with the experimentally observed strength cdfs of structures made of various quasi-brittle materials such as composites and engineering and dental ceramics (Bažant and Pang 2007; Bažant et al. 2009; Le et al. 2011). In the meantime, the finite weakest-link model leads to a non-Weibullian size effect on the mean structural strength, which matches the predictions of other mechanical models such as the cohesive crack and nonlocal models.

The energetic scaling theory applies to structures that contain a large preexisting stress-free crack formed prior to the maximum load. The existence of the stress-free crack causes significant stress concentration so that the crack growth is guaranteed to emanate from its front. Therefore, whatever randomness exists in material strength will not affect the scaling of mean strength. Extensive studies have shown that an approximate form of this size effect, which is in agreement with experiments on concrete, ceramics, composites, and sea ice, can be derived from the equivalent linear elastic fracture mechanics based on the asymptotic expansion of the energy release rate function or the J-integral (Bažant 1984, 2004, 2005; Bažant and Kazemi 1990; Bažant and Planas 1998). In contrast with the aforementioned statistical scaling theory, this type of size effect is usually referred to as the energetic (or deterministic) scaling. A recent study also investigated the energetic scaling of fracture of structures with a reentrant corner with a sharp angle under Mode-I loading (Bažant and Yu 2006).

It is clear that the statistical and energetic scaling theories represent two limiting cases: (1) statistical scaling theory is limited to structures without stress singularities, where in principle the damage could initiate at any location within the structure; and (2) energetic scaling theory applies to structures with relatively strong stress singularities, where the location of damage is predetermined. This indicates that the type of scaling mechanism depends on the order of stress singularities of the structure. As the stress singularities of the structure diminish, the energetic scaling should transition to the statistical scaling. However, it is not clear how the scaling mechanism transits between these two limiting cases. Understanding such a transitional scaling behavior is important, because many modern engineering structures often contain geometric discontinuities that produce weak stress singularities. The simplest case is a homogeneous structure with a reentrant corner, i.e., a V-notch, where the notch angle governs the order of the stress singularity. This study aims to formulate a universal size effect equation for quasi-brittle structures that bridges the existing statistical and energetic scaling theories through theoretical and numerical investigations on the fracture of structures with a V-notch under Mode-I loading.

## Review of Energetic and Statistical Scaling Theories

Consider a homogeneous structure of positive geometry containing a V-notch and subjected to Mode-I loading (Fig. 1), where the notch angle is denoted by $\gamma$. Positive geometry, which is typical of most structural geometries, is defined such that the peak load is reached once the fracture process zone (FPZ) is fully developed (Bažant 2005). In this paper, the authors also assume that the notch is sufficiently deep, i.e., $\alpha = a/D > 0.1$, where $a$ is the notch depth and $D$ is the depth of the structure. In general, the stress concentration at the V-notch tip is governed by two distinct stress singularities, which represent the symmetrical and antisymmetrical deformation modes (Williams 1952; Carpenter 1984; Sinclair et al. 1984; Liu et al. 1998). For Mode-I fracture, only one stress singularity $\lambda$ prevails, which corresponds to displacements that are symmetric about the centerline of the notch. This section reviews the existing energetic and statistical scaling theories for the two limiting cases.

### Case of Strong Stress Singularity

If the stress singularity is sufficiently strong, then the FPZ must form at the notch tip. Therefore, the corresponding scaling mechanism is deterministic in nature. The fracture of the V-notch has been analyzed both theoretically and experimentally (Ritchie et al. 1973; Carpinteri 1987; Seweryn 1994; Dunn et al. 1996, 1997; Gomez and Elices 2003). Three commonly used fracture criteria are as follows: (1) the peak load is reached when the stress intensity factor reaches a critical value (Carpinteri 1987; Dunn et al. 1996; Gomez and Elices 2003), (2) the peak load is attained when the stress at a certain distance $c_f$ from the notch tip reaches the material tensile strength (Ritchie et al. 1973; Bažant and Yu 2006), and (3) the peak load is realized when the energy release rate of an equivalent crack that represents the FPZ reaches a critical value, i.e., the fracture energy (Le et al. 2010; Le 2011). These criteria are all cast in the framework of linear elastic fracture mechanics, which can be used to derive the large-size asymptote of the size effect. For Mode-I loading, all three criteria essentially yield the same form of the power-law size effect. Among these criteria, the second criterion is relatively straightforward to use, because the first criterion adopts a geometry-dependent critical stress intensity factor, which needs to be measured for every notch angle (Bažant and Yu 2006), and the third criterion requires determination of the energy release at the tip of an equivalent crack through the solution of an ancillary boundary value problem (Leguillon 2002; Le et al. 2010; Le 2011).

Based on the Williams solution, the stress field near the notch tip under Mode-I loading can be expressed as

$$\sigma_{ij} = Hr^{\lambda} f_{ij}(\theta, \gamma) \tag{1}$$

where $r$ = radial distance from the notch tip; $f_{ij}(\theta, \gamma)$ = dimensionless function describing the angular dependence of the stress; $\lambda$ = order of stress singularity; and $H$ = stress intensity factor. Dimensional analysis allows the stress intensity factor $H$ to be written as

$$H = \sigma D^{-\lambda} h(\gamma) \tag{2}$$

where $\sigma$ = nominal stress = $cP/bD$; $P$ = applied load; $c$ = constant; $b$ = width of the structure in the transverse direction; and $h(\gamma)$ = dimensionless stress intensity factor, which depends on the geometry of the structure. Considering the second failure criterion, the nominal strength can be calculated as

$$\sigma_N = f_t' \psi(\gamma) \left(D/c_f\right)^{\lambda} \tag{3}$$

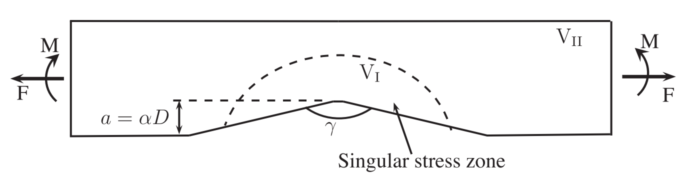


**Fig. 1.** Structure with a V-notch under Mode-I fracture

where $\psi(\gamma) = h^{-1}(\gamma)\ f_{\theta\theta}^{-1}(0,\gamma)$; and $f_t'$ = tensile strength. Eq. (3) represents the large-size asymptote of the energetic size effect law. The small-size asymptote is relatively easy to obtain, because for small-size structures, the FPZ occupies the entire crack ligament. Consequently, the ligament must behave like a crack filled with plastic glue. At this plastic limit, the size effect must vanish. An approximate equation that bridges these two asymptotes has been proposed (Bažant and Yu 2006; Le 2011)

$$\sigma_N = \sigma_s \left[1 + \left(D/D_{0\gamma}\right)^{1/\beta_\gamma}\right]^{\lambda\beta_\gamma} \tag{4}$$

where $\sigma_s$ = nominal strength at the small-size limit; $\beta_\gamma$ = model parameter; and $D_{0\gamma}$ = transitional size. Eq. (4) clearly indicates the effect of the stress singularity on the scaling law for the case where the stress singularity is sufficiently strong. When $\lambda = -1/2$ and $\beta_\gamma = 1$, Eq. (4) represents the classical Type-2 size effect (Bažant 1984, 2004), which applies to structures with a large preexisting crack.

### Case of Zero-Stress Singularity

The limiting case of the zero-stress singularity corresponds to unnotched structures, for which it is uncertain where damage initiates and localizes. The structure reaches its peak load once any one of the RVEs is damaged, and thus, the RVE is defined as the smallest material volume whose failure triggers the failure of the entire structure. The size of the RVE is approximately two to three times the size of material inhomogeneities. Statistically speaking, the structure can be represented by a chain of RVEs. Because the RVE size is about the same as the autocorrelation length of the random strength field (Grassl and Bažant 2009), the RVE strength can be treated as an independent random variable, and the failure probability of the structure can then be written as

$$P_f(\sigma_N) = 1 - \prod_{i=1}^{n} \{1 - P_1[\sigma_N s(\mathbf{x}_i)]\} \tag{5}$$

where $P_1$ = cdf of RVE strength; $n$ = number of RVEs in the structure; and $s(\mathbf{x}_i)$ = dimensionless stress field such that $\sigma_N s(\mathbf{x}_i)$ = maximum elastic principal stress at the center of the $i$th RVE with a coordinate $\mathbf{x}_i$.

Based on atomistic fracture mechanics and a statistical multiscale transition model, it has been shown that the cdf of RVE strength can be approximated by a Gaussian distribution grafted by a Weibull tail at a probability within the range of $10^{-4}$–$10^{-3}$ (Bažant et al. 2009; Le et al. 2011), i.e.

$$P_1(\sigma) = 1 - \exp\left[-\langle\sigma\rangle^m / s_0^m\right] \qquad (\sigma < \sigma_{gr}) \tag{6}$$

$$P_1(\sigma) = P_{gr} + \frac{r_f}{\sqrt{2\pi}\delta_G} \int_{\sigma_{gr}}^{\sigma} e^{-(\sigma'-\mu_G)^2/2\delta_G^2} \mathrm{d}\sigma' \quad (\sigma \geq \sigma_{gr}) \tag{7}$$

where $\langle x\rangle = \max(x, 0)$; $\mu_G$ and $\delta_G$ = mean and standard deviation of the Gaussian core if considered extended to $-\infty$; $s_0$ and $m$ = scale and shape parameters of the Weibull tail ($m$ is also called the Weibull modulus); $r_f$ = scaling parameter required to normalize the grafted cdf such that $P_1(\infty) = 1$; $P_{gr}$ = grafting probability $= 1 - \exp[-(\sigma_{gr}/s_0)^m]$, and the continuity of the probability density function at the grafting stress requires that $(\mathrm{d}P_1/\mathrm{d}\sigma)|_{\sigma_{gr}^+} = (\mathrm{d}P_1/\mathrm{d}\sigma)|_{\sigma_{gr}^-}$.

The mean strength of the structure can be calculated as $\overline{\sigma}_N = \int_0^1 \sigma_N \mathrm{d}P_f = \int_0^\infty [1 - P_f(\sigma_N)]\mathrm{d}\sigma_N$. By considering structures of different sizes, the authors readily obtain the size effect on the mean structural strength. Though it is impossible to derive a closed-form expression, an approximate equation for this size effect has been proposed, which is referred as the Type-1 size effect (Bažant and Novák 2000; Bažant 2004, 2005)

$$\overline{\sigma}_N = \left[\frac{N_a}{D} + \left(\frac{N_b}{D}\right)^{rn_d/m}\right]^{1/r} \tag{8}$$

where $n_d$ = dimension of scaling; and $N_a$, $N_b$, and $r$ = constants, which can be directly related to the statistical parameters of the cdf of RVE strength, i.e., $\mu_G$, $\delta_G$, and $s_0$, based on the small-size and large-size asymptotes (Cannone Falchetto et al. 2013). It should be pointed out that Eq. (8) does not apply as $D \to 0$, which makes sense because the continuum model breaks down for structures of very small size. A recent study (Le et al. 2012) has shown that for small and intermediate-size structures the size effect derived from this finite weakest-link model with the use of elastic stresses agrees well with the prediction from the nonlinear deterministic calculation. This is because the mean size effect behavior for small-size and intermediate-size structures is mainly caused by the operative stress redistribution mechanism, which can be well predicted by a nonlinear deterministic calculation. At the same time, this mechanism can also be captured by the finite weakest-link model, where the statistical multiscale transition model used for the formulation of the cdf of RVE strength consists of statistical bundles and chains that represent the damage localization and load redistribution mechanisms at different scales (albeit only the elastic stresses are used) (Le et al. 2011, 2012). For large-size structures, the zone of stress redistribution is negligible compared with the structure size, and the size effect is mainly caused by the randomness of material strength, which cannot be captured by the deterministic calculation. Therefore, the size effect curve for the case of the zero-stress singularity can be completely explained by the finite weakest-link model.

## Generalized Weakest-Link Model for Transitional Scaling Behavior

For structures with a wide V-notch and therefore a weak stress singularity, there is no guarantee that the FPZ would form at the tip of the V-notch. This means that the failure of the structure can be statistically represented by the weakest-link model. On the other hand, there exists a singular stress field at the V-notch tip even though the degree of stress concentration is not significant. Furthermore, the fracture of the V-notch itself is associated with an energetic scaling law shown as Eq. (4), which cannot be represented by the existing finite weakest-link model. This prompts us to derive a new scaling model by generalizing the classical finite weakest-link model to include the energetic scaling of fracture of the V-notch.

In the proposed generalized weakest-link model, the singular stress zone is isolated from the remaining part of the structure (Fig. 1), where the singular stress zone can be determined by comparing Eq. (1) to the numerically simulated elastic stress field. Because the singular stress zone is influenced by the presence of the V-notch, whose fracture exhibits an energetic scaling [i.e., Eq. (4)], the authors propose to include this energetic scaling for the calculation of the failure probability of the singular stress zone

$$P_{f,V_I}(\sigma_N) = 1 - \prod_{i=1}^{N_1} \{1 - P_1[\mu(D)\sigma_N s(\mathbf{x}_i)]\} \tag{9}$$

$$\mu(D) = \left[1 + \left(D/D_{0\gamma}\right)^{1/\beta_\gamma}\right]^{-\lambda\beta_\gamma} \tag{10}$$

where the parameters in the scaling term $\mu(D)$ follow the same definitions as those in Eq. (4); and $N_1$ = number of RVEs in the singular stress zone. For the remaining part of the structure, a conventional weakest-link model can be used, i.e.

$$P_{f,V_{II}}(\sigma_N) = 1 - \prod_{i=1}^{N_2}\{1 - P_1[\sigma_N s(\mathbf{x}_i)]\} \tag{11}$$

where $N_2$ = number of RVEs in the region outside the singular stress zone. Therefore, the failure probability of the entire structure can easily be written as

$$P_f = 1 - \left(1 - P_{f,V_I}\right)\left(1 - P_{f,V_{II}}\right) \tag{12}$$

from which the mean structural strength can be calculated. Similar to the conventional weakest-link model, a closed-form solution is not expected. In this paper, the authors seek an approximate scaling equation through asymptotic matching.

At the large-size limit, the failure probability of the structure is governed by the tail part of the strength cdf of one RVE. Based on the fact that $\ln(1-x) \approx -x$ for $x \to 0$, the weakest-link model for 2D structures can be rewritten as

$$P_f(\sigma_N) = 1 - \exp\left[-\int_{V_I} \frac{\mu^m(D)\sigma_N^m \langle s(\mathbf{x})\rangle^m}{s_0^m}\frac{\mathrm{d}V(\mathbf{x})}{l_0^2} - \int_{V_{II}} \frac{\sigma_N^m \langle s(\mathbf{x})\rangle^m}{s_0^m}\frac{\mathrm{d}V(\mathbf{x})}{l_0^2}\right] \tag{13}$$

where $l_0$ = RVE size. Because $s(\mathbf{x})$ represents the normalized elastic stress field, the linear transformation of the coordinate, i.e., $\xi = \mathbf{x}/D$, can be used to rewrite Eq. (13) as

$$P_f(\sigma_N) = 1 - \exp\left\{-\left[\mu^m(D)\Psi_1 + \Psi_2\right]\frac{D^2}{l_0^2}\left(\frac{\sigma_N}{s_0}\right)^m\right\} \tag{14}$$

$$\Psi_1 = \int_{V_I} \langle s(\mathbf{x})\rangle^m \mathrm{d}V(\xi), \quad \Psi_2 = \int_{V_{II}} \langle s(\mathbf{x})\rangle^m \mathrm{d}V(\xi) \tag{15}$$

Based on Eq. (14), the mean strength can easily be calculated

$$\overline{\sigma}_N = s_0\left[\mu^m(D)\Psi_1 + \Psi_2\right]^{-1/m}\Gamma\left(1 + \frac{1}{m}\right)\left(\frac{l_0}{D}\right)^{2/m} \tag{16}$$

where $\Gamma(x)$ = Gamma function. Eq. (16) indicates that the large-size asymptote of the size effect curve differs from the classical Weibull size effect because of the fact that the authors introduced the energetic scaling term for the failure statistics of the singular stress zone. It should be pointed out that the authors use the integral form for the weakest-link model. In principle, $\Psi_1$ is infinite because of the singular stress field. However, in the spirit of the weakest-link model, the singular stress at the V-notch tip should not be included (Bažant and Xi 1991; Bažant et al. 2010). Therefore, when evaluating $\Psi_1$, the authors exclude the region where the radial distance from the notch tip is less than a certain distance $d_c$. As will be discussed later, the choice of $d_c$ is not particularly important for the present formulation.

At the small-size limit, the entire structure consists of a very small number of RVEs. Therefore, it is expected that the RVEs in the singular stress zone govern the failure of the entire structure. For structures without stress singularities, it has been shown that the small-size asymptote of the size effect can be expressed as $\overline{\sigma}_N \propto (D/D_b)^{-1/r}$, where $D_b$ and $r$ can be determined from the mean and standard deviation of the Gaussian part of the cdf of RVE strength (Cannone Falchetto et al. 2013). In the present model, the stress that governs the failure of the singular stress zone is scaled by the energetic scaling term. Therefore, the corresponding size effect at the small-size limit can be expressed as $\overline{\sigma}_N \propto \mu^{-1}(D)(D/D_b)^{-1/r}$.

Because the present model is developed within the framework of the weakest-link model, the entire size effect curve can be approximated by a function similar to Eq. (8). Meanwhile, it is also clear that, as the stress singularity is sufficiently strong, all the scaling terms associated with the weakest-link model should vanish. This transition is expected to occur in a very narrow range of stress singularities, which is in this paper approximated by a Gaussian function. Therefore, the authors propose the following scaling equation, which bridges the limiting cases of strong and zero-stress singularities:

$$\overline{\sigma}_N = \sigma_0\left\{C_1[\mu^m(D)\Psi_1 + \Psi_2]^{-r/m}\left(\frac{D + l_s}{l_0}\right)^{-2r/m} \times \exp\left[-(\lambda/\lambda_1)^2\right] + \frac{\mu^{-r}(D)D_b}{\exp\left[-(\lambda/\lambda_2)^2\right]D + l_p}\right\}^{1/r} \tag{17}$$

where $\sigma_0$ = reference stress; and $C_1$, $r$, $\lambda_1$, $\lambda_2$, $l_s$, $l_p$, and $D_b$ = constants. Note that, slightly different from the form of Eq. (8), the authors purposely introduce constants $l_s$ and $l_p$ to regularize the functional behavior as $D \to 0$. The large-size and small-size asymptotes yield

$$\sigma_0 C_1^{1/r} = s_0^{1/r}\Gamma(1 + 1/m) \tag{18}$$

$$\sigma_s = \sigma_0\left\{D_b/l_p + C_1(\Psi_1 + \Psi_2)^{1/m}(l_s/l_0)^{-2r/m} \times \exp\left[-(\lambda/\lambda_1)^2\right]\right\}^{1/r} \tag{19}$$

The small-size strength limit $\sigma_s$ ($D \to 0$) can usually be obtained by simple plastic analysis, where the ligament is considered to be filled up with a plastic glue.

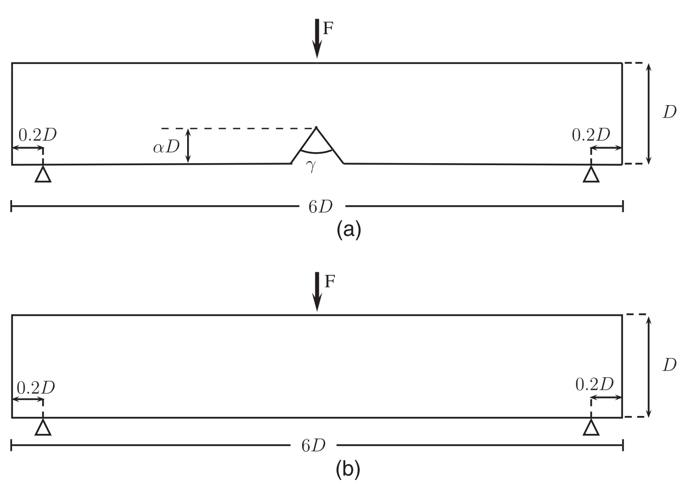


**Fig. 2.** Numerical simulation of three-point bend beams

It is clear that Eq. (17) converges to Eqs. (4) and (8) as the two limiting cases. For the transition between these two limits, the size effect consists of both energetic and statistical components. At the small-size limit, the size effect is mainly governed by the statistical scaling component, because the energetic scaling term predicts a weak size effect. At the large-size limit, the scaling is governed by the Weibull statistics modified by an energetic scaling term, which leads to a compound energetic-Weibull statistical scaling. The detailed calibration of Eq. (17) will be presented in the "Numerical Simulation" section.

Though the focus of this study is on Mode-I fracture, the present framework can be readily extended to general mixed-mode fracture (Le 2011; Le and Xue 2013), which is applicable to bimaterial structures. When dealing with mixed-mode fracture, the energetic term would contain two distinct stress singularities, and the large-size asymptote of the energetic size effect has to be derived from an energetic argument (Le 2011).

## Numerical Simulation

### *Model Description*

To verify the proposed analytical model, the authors investigate the size effect on the strength of concrete beams with a V-notch under

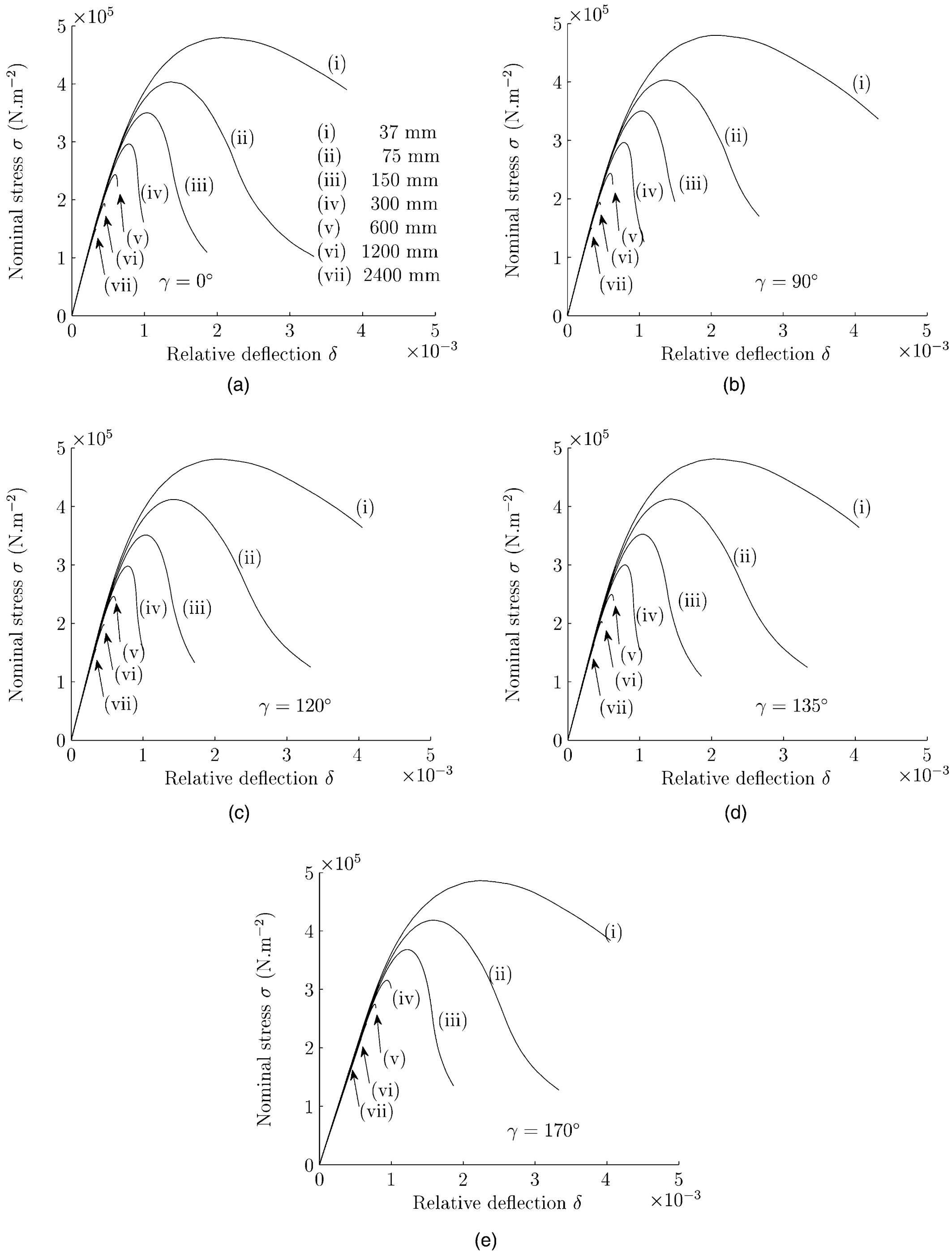


**Fig. 3.** Simulated nominal stress–relative displacement curves

three-point bending [Fig. 2(a)]. The beam has a 6:1 span-to-depth ratio, and the notch depth is 20% of the beam depth. In the simulation, the authors consider five different notch angles, i.e., $\gamma = 0$, 90, 120, 135, and 170°, and a series of geometrically similar specimens with a size range 1:2:4:8:16:32:64:128. (The depths of the smallest and largest specimens are 37.5 mm and 4.8 m, respectively.) Based on the Williams solution, these notch angles correspond to the following orders of Mode-I stress singularity: $\lambda = -0.5$, $-0.4555$, $-0.3843$, $-0.3264$, and $-0.0916$. In addition to this set of specimens, the authors also include the size effect simulation of flat beams with a maximum size ratio 1:64 [Fig. 2(b)], which corresponds to the case of zero-stress singularity.

It is well known that concrete exhibits a complex constitutive behavior. Extensive efforts have been devoted to numerical modeling of the fracture of concrete (Mazars 1986; Lubliner et al. 1989; Lee and Fenves 1998; Jirásek and Zimmermann 1998; Bažant et al. 2000). Because the authors are interested in static Mode-I fracture, they adopt the default plastic-damage model in *ABAQUS 6.11,* because it is sufficient for the purpose of the current study; a detailed description of this constitutive model can be recovered from Dassault Systèmes Simulia (2011). The material properties are chosen as follows: Young's modulus $E = 30$ GPa, Poisson ratio $\nu = 0.2$, tensile strength $f_t' = 3$ MPa, compressive strength $f_c' = 30$ MPa, and Mode-I fracture toughness $G_f = 100\ \mathrm{N \cdot m^{-1}}$. Though the authors specify the compressive strength, the compressive region of the beam is expected to remain elastic. Therefore, the nonlinear part of the compressive behavior is not of particular interest for the current study. All specimens undergo displacement-controlled loading.

In this study, the numerical simulation is performed within a deterministic framework. Previous studies have shown that the deterministic simulation with a strain-softening constitutive model can successfully capture the entire size effect for the case of strong stress singularity and the size effect for the small and intermediate structure sizes for the case of zero-stress singularity (Bažant 2004, 2005; Bažant et al. 2007). Therefore, the authors expect that for the case of weak stress singularity the deterministic numerical model is sufficient for simulating the size effect for the small and intermediate structure sizes. For the large-size asymptote, the deterministic simulation cannot yield the Weibull statistical scaling. In this study, the authors mainly focus on the small- and intermediate-size range, which is applicable to most engineering designs. Therefore, only deterministic simulation is necessary. As will be shown later, the influence of the Weibull scaling component only prevails in structures of very large size.

In the finite-element modeling, the notch tip is considered to have a very small width, i.e., 5 mm, which is a constant for all the geometries and sizes. For the deterministic simulation, the damage occurs near the midspan of the beam. Therefore, to reduce computational efforts, the authors model the middle portion of the beam with a refined mesh (i.e., 5 mm) and the damage plasticity model, whereas the rest part of the beam is modeled by a coarse mesh with a purely elastic model. For each specimen, the assumed region where the nonlinear material model is used is further checked as part of the simulation. As the notch angle increases, this nonlinear region becomes larger. It should be pointed out that the present modeling is not as efficient as the crack band model and the nonlocal model, where larger element sizes can be used. However, the use of the crack band and nonlocal models requires extensive modeling efforts with special cautions such as the choice of the crack band width (Bažant and Planas 1998; Jirásek and Bauer 2012) and treatment of the nonlocal weighting function along the structural boundary (Bažant et al. 2010).

## Results and Discussion

Fig. 3 presents the simulated nominal stress–relative deflection curves for specimens of all sizes and all different notch angles, where the nominal stress is defined as $\sigma = P/bD$ and the relative

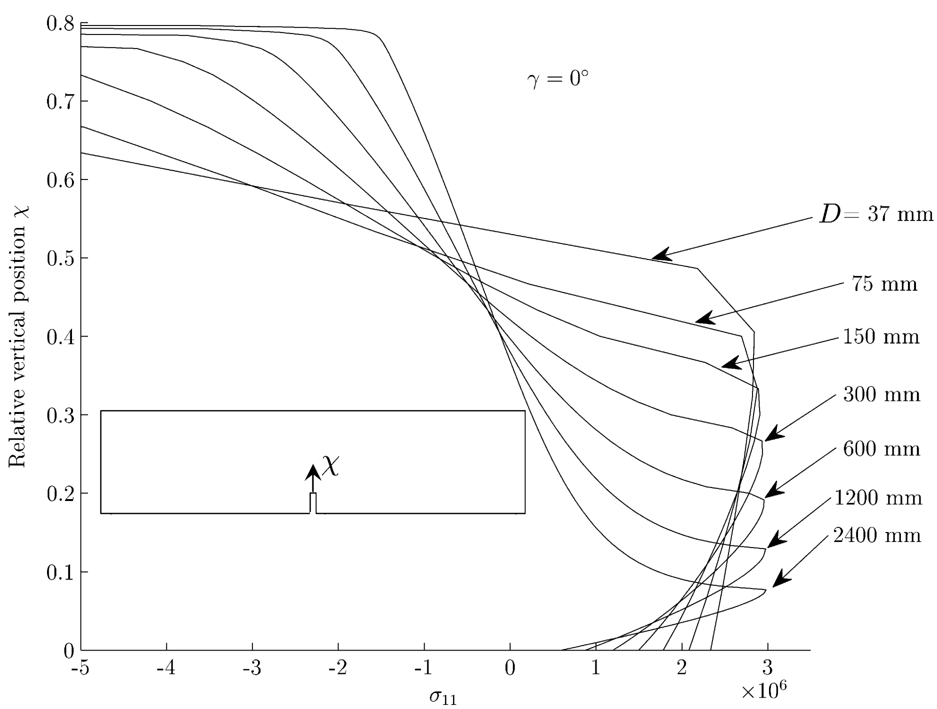


**Fig. 4.** Profile of normal stress along the ligament

displacement is defined as $\delta = \Delta/D$ ($\Delta$ is the load-point displacement). It is observed that, as the structure size increases, the postpeak softening portion of the load-deflection curve becomes steeper, which implies a more brittle failure behavior. It should be noted that for large specimens (i.e., $D \geq 1.2$ m) the postpeak behavior is not captured, which indicates that a snap-back instability may have occurred. It is noted that the snap-back behavior could be captured by loading the specimens by the crack mouth opening displacement. This is not done, because the authors are interested only in the peak load. The size-dependent failure behavior can also be explained by the nonlinear fracture process. Fig. 4 shows the normal stress profile along the notch ligament at the peak load for beams with a sharp notch (i.e., $\gamma = 0°$). Similar stress profiles are observed for beams with other notch angles. The FPZ can be determined as the region where a strain-softening behavior occurs. As seen, at the large-size limit, the FPZ becomes negligible compared with the structure size,

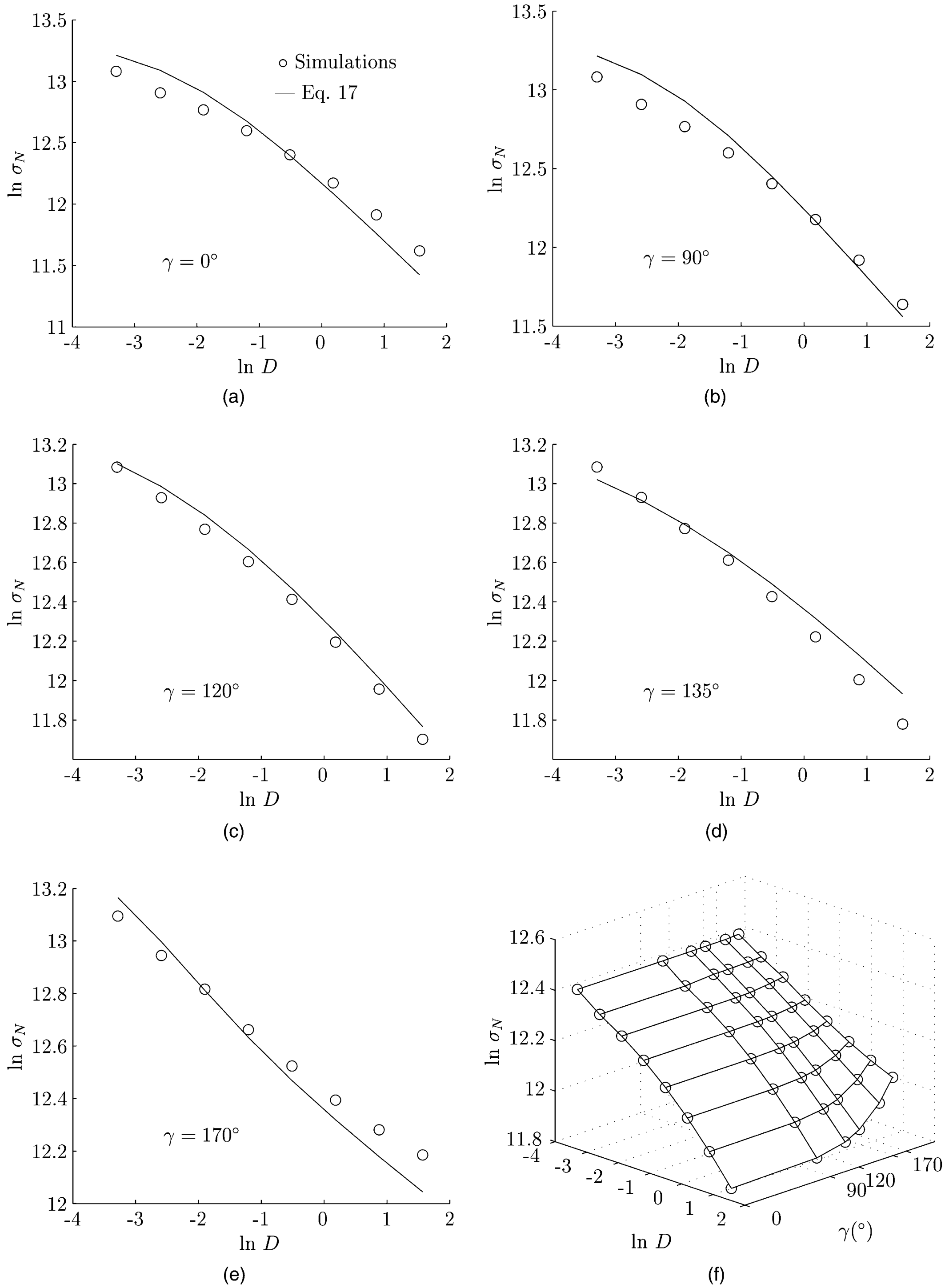


**Fig. 5.** Size effect curves for notched beams: (a)–(e) simulated size effect curves with optimum fits by Eq. (17) for individual notch angles; (f) 3D plot of simulated effects of structural size and notch angle on nominal strength

whereas at the small-size limit the FPZ occupies a large part of the structure and the stress in the FPZ exhibits more or less a plastic profile. This implies that the structure would behave in a quasi-plastic manner at the small-size limit and in a brittle manner at the large-size limit, which is consistent with the conclusion drawn from the postpeak behavior of the load-deflection curves.

For the 2D specimens studied, the authors define the nominal strength of the beam simply as $\sigma_N = P_{\max}/bD$, where $b = 1$. Fig. 5 presents the simulated size effects on the nominal strength for different notch angles, and Fig. 6 presents the simulated size effect curve for specimens without notches. At the small-size limit, the nominal strengths of all the specimens are almost the same, whereas the large-size asymptotes of the size effect vary with different notch angles [Fig. 5(f)]. As the notch angle increases, which implies that the stress singularity becomes weaker, there is a clear change in the curvature of the size effect curve.

The simulated size effect curves are now compared with the proposed approximate size effect equation. As mentioned earlier, Eq. (17) contains the energetic and statistical scaling terms. For the energetic scaling term $\mu(D)$, previous study (Bažant and Yu 2006) has shown that parameter $D_{0\gamma}$ varies with the notch angle as $D_{0\gamma} = D_0\psi(\gamma)/\psi(0)$, where $D_0 = D_{0\gamma}$ at $\gamma = 0$. Elastic analysis directly yields $\psi = 2, 1.836, 1.624, 1.485, \text{and } 1.089$ for $\gamma = 0, 90, 120,$ 135, and 170°. The parameter $\beta_\gamma$ is introduced for a better fitting of size effect data. Therefore, in principle, $\beta_\gamma$ may vary with the notch angle. For the case of zero notch angle, the classical Type-2 size effect indicates that $\beta_\gamma$ should be equal to 1. However, for other notch angles, $\beta_\gamma$ may take other positive values. In this study, the authors leave $\beta_\gamma$ as a calibration constant for every notch angle except for the zero notch angle.

For the statistical scaling terms, constants $\Psi_1$ and $\Psi_2$ for the Weibullian part can be easily determined by linear elastic analysis. In this paper, the authors assume that the RVE size $l_0$ is equal to 37.5 mm, which is about three times the size of the typical maximum aggregate $d_a$ (Bažant and Pang 2007). Furthermore, when calculating $\Psi_1$, the authors exclude the notch tip region with a radius $d_c = D/400$ to avoid an infinite value of $\Psi_1$. It should be noted that the authors only need $d_c$ for the size effect curves for the case of weak stress singularity. The choice of $d_c$ is not particularly important, because the authors further introduced two Gaussian functions to describe this transitional scaling mechanism, and different values of $d_c$ will be compensated by function $\exp[-(\lambda/\lambda_1)^2]$, which is empirically determined by the optimum fit of the data. Other parameters associated with the statistical scaling components can be determined based on the size effect curve of the unnotched beam. The Weibull modulus $m$ for concrete is known to be 24 (Bažant and Novák 2000; Bažant and Pang 2007). Previous studies have shown that $D_b$ is approximately equal to $4d_a$ (Bažant 2005), which is about 50 mm for this study. As shown in Fig. 6, $r$, $l_p$, $l_s$, and $s_0$ can be determined by the optimum fitting of the size effect curve of the flat beam.

The reference stress $\sigma_0$ can then be determined based on the nominal strength at the small-size limit. As mentioned earlier, the structural strength at the small-size limit can be calculated by a plastic analysis, where it can be assumed that the FPZ behaves in a plastic manner as shown in Fig. 7. The stress profile of the FPZ (Fig. 4) further verifies such a plastic model. Therefore, regardless of the notch angle, the nominal strength at the small-size limit can be calculated as $\sigma_s = 16/75\, f_t' = 0.64$ MPa. Similarly, it can easily be shown that $\sigma_s$ for the unnotched beams considered in the simulation is equal to 1 MPa. With Eq. (19), the authors can calculate $\sigma_0$, which is expected to vary with the notch angles, and Eq. (18) yields constant $C_1$. The last two parameters $\lambda_1$ and $\lambda_2$ can be determined by fitting the simulated size effect curve for the case of weak stress singularities.

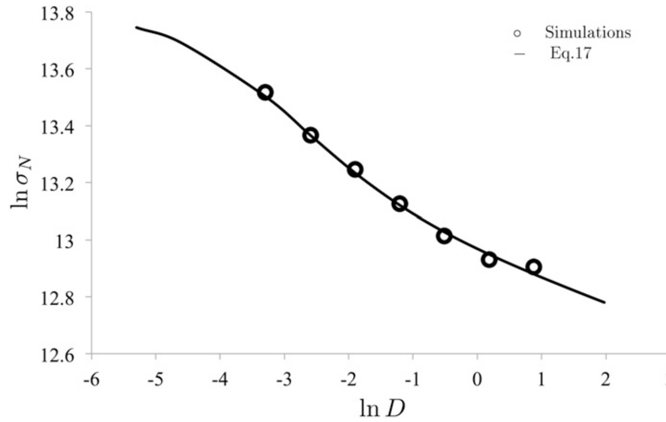


**Fig. 6.** Simulated size effect curves for unnotched beams with optimum fits by Eq. (17)

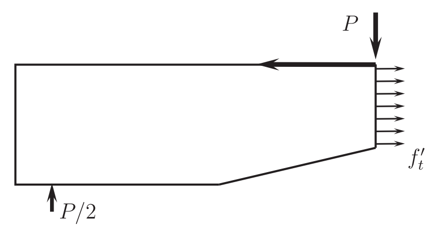


**Fig. 7.** Plastic analysis at small-size limit

The aforementioned procedure is now applied to calibrate Eq. (17) from the simulated size effect curves. By fitting the size effect curve for the case of an unnotched beam, the authors obtain $l_p = 40$ mm, $l_s = 150$ mm, $r = 0.88$, and $s_0 = 0.448$ MPa. Based on the size effect curves for the series of V-notch specimens, $D_0 = 90$ mm, $\lambda_1 = 0.301$, $\lambda_2 = 0.208$, and $\beta_\gamma = 1, 1, 1.5, 2, \text{and } 1.1$ for $\gamma = 0, 90, 120, 135, \text{and } 170°$. Figs. 5 and 6 show that the simulated size effect curves can be well fitted by Eq. (17). It should be pointed out that the simulated size effect curve does not match well with Eq. (17) at the large-size limit for beams with a 170° V-notch and unnotched beams. This is because of the fact that deterministic simulation was used, which cannot capture the associated large-size asymptote of the classical Weibull scaling relation. Furthermore, it is observed that such a difference occurs for very large beam size (i.e., $D \geq 1.2$ m), which indicates that deterministic calculation is sufficient for most normal-size concrete beams.

It should be emphasized that the proposed size effect equation is limited to the case of deep notches. If the notch depth becomes shallow (for instance $a/D < 0.05$), it is expected that the statistical scaling components will play a dominant role regardless of the notch angle, which represents another type of transition between the energetic and statistical scaling mechanisms. Fig. 8 presents the numerically simulated size effect curve for different $\alpha$-values (i.e., $\alpha = 0, 0.05, 0.1, \text{and } 0.2$) for various notch angles, which clearly demonstrates such a transition. This transition behavior has been analytically and experimentally studied for structures with a preexisting crack (Bažant and Yu 2009; Zegeye et al. 2012; Hoover et al. 2013). Nevertheless, a general scaling law that accounts for both effects of stress singularities and notch depth is still to be explored.

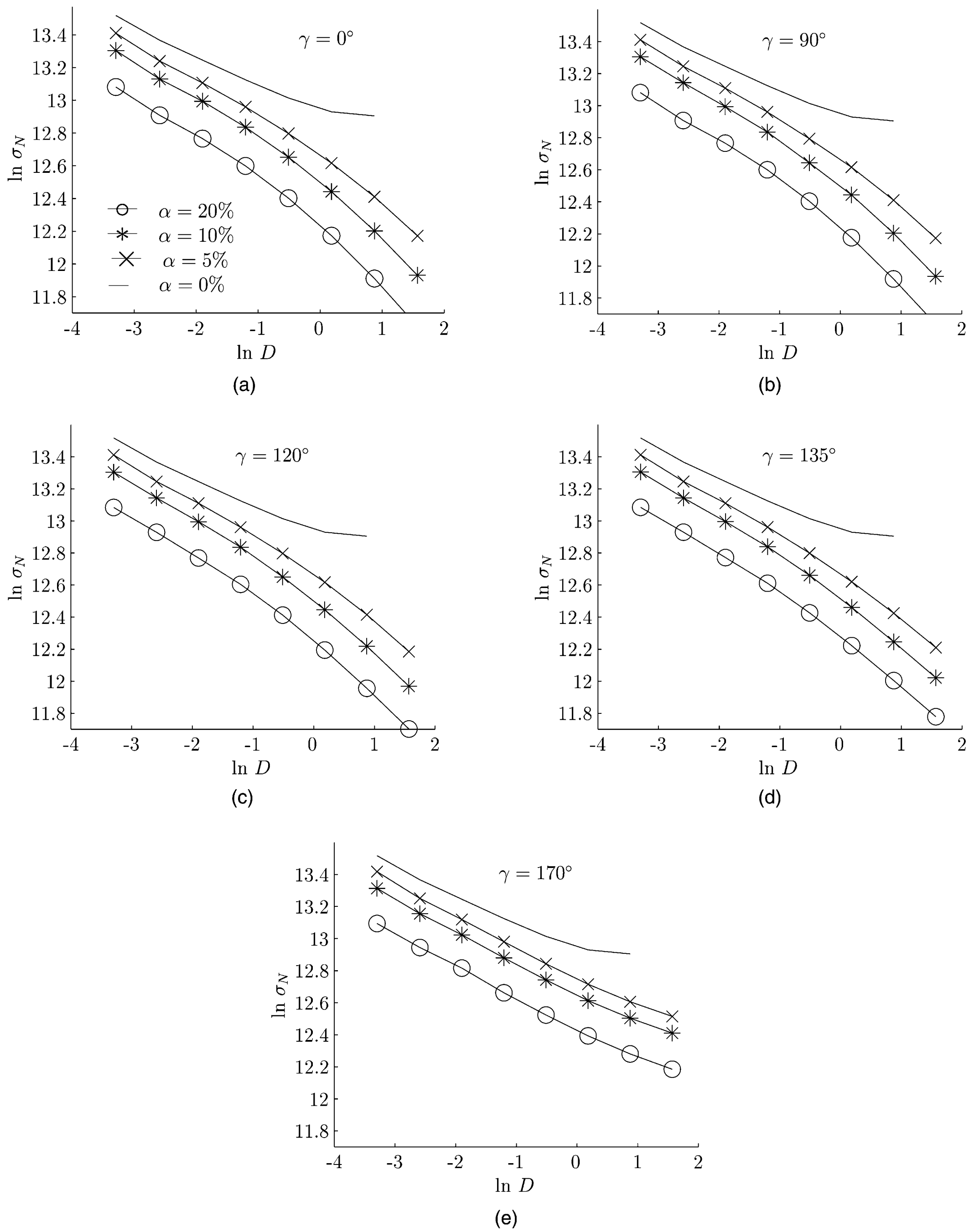


**Fig. 8.** Effect of notch depth on size effects

## Conclusions

This study shows that the scaling of quasi-brittle fracture strongly depends on the stress singularities of the structure. Such dependence can be derived from a generalized weakest-link model, where the classical energetic scaling of fracture is incorporated into the classical finite weakest-link model. For the case of strong stress singularity, the scaling of structural strength is purely energetic, which can be obtained from fracture mechanics. For the case of zero-stress singularity, the size effect can be explained by the random material strength through the finite weakest-link model. For the case of weak stress singularity, the scaling of fracture is governed by both energetic and statistical mechanisms. The present computational study indicates that for small- and intermediate-size structures the scaling behavior can alternatively be captured by nonlinear deterministic simulations regardless of the order of stress singularities.

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