# A Long Crack Penetrating a Transforming Inhomogeneity

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*This note presents the stress intensity factors of a long crack penetrating a circular transforming inhomogeneity. Using the Greens functions of dislocations interacting with a circular inhomogeneity experiencing an isotropic (free expansion) eigenstrain, the elasticity solution is reduced to a system of singular integral equations representing the traction boundary condition along the crack surfaces. The normalized stress intensity factor, obtained through a numerical solution of the integral equations, has a strong dependence on the elastic mismatch, and can be either negative or positive depending on the crack-tip location. The formulation and results generalize a previously published transformation-toughening model that assigns equal elastic moduli to the inhomogeneity and the surrounding medium.*
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## Analysis

Consider the plane elastostatics problem shown in Fig. 1(*a*). A circular inhomogeneity with radius *a* Poisson's ratio $\nu_2$, and shear modulus $\mu_2$, is embedded in an infinite plate with Poisson's ratio $\nu_1$ and shear modulus $\mu_1$. A semi-infinite crack penetrates the inhomogeneity, which is experiencing an isotropic (free expansion) eigenstrain, $\varepsilon_{ij}^* = \delta_{ij} e^*$, where the $\varepsilon_{ij}$ are the components of the strain tensor and $\delta_{ij}$ is the Kronecker delta. The bonding between the inhomogeneity and the surrounding matrix is perfect. As shown in Fig. 1(*a*), the origin of the coordinate system is located at the center of the inhomogeneity, and the crack tip is located at point $(w,0)$. The stress intensity factor produced by the eigenstrain within the inhomogeneity is defined as $K_I^{\mathrm{loc}}$.

The solution is formulated as the superposition of two problems, as shown schematically in Figs. 1(*b*) and 1(*c*). The first involves the stresses produced along the crack line in an uncracked plate containing the expanding inhomogeneity (Fig. 1(*b*)), and the second the stresses produced along the crack line by a continuous distribution of dislocations (Fig. 1(*c*)). The stresses produced by the eigenstrain are

$$\sigma_{yy}^{1e} = \frac{4(1+\eta)\mu_1\mu_2 e^*}{\mu_1(\kappa_2-1)+2\mu_2}\left\{\frac{a^2}{x^2}\right\} \quad x \leqslant a \tag{1a}$$

$$\sigma_{yy}^{2e} = \frac{4(1+\eta)\mu_1\mu_2 e^*}{\mu_1(\kappa_2-1)+2\mu_2} \quad x \geqslant a \tag{1b}$$

while those resulting from the distribution of dislocations are represented as

$$\sigma_{yy}^{1d} = \int_0^\infty \frac{2b_1(t)}{1-x}dt + \int_a^\infty K_{11}(x,t)b_1(t)dt + \int_w^a K_{12}(x,t)b_2(t)dt \quad x \leqslant a \tag{2a}$$

$$\sigma_{yy}^{2d} = \int_w^a \frac{2b_2(t)}{t-x}dt + \int_w^a K_{21}(x,t)b_2(t)dt + \int_a^\infty K_{22}(x,t)b_1(t)dt \quad w \leqslant x \leqslant a. \tag{2b}$$

In Eq. (2), $b_i(t) = \mu_i/\pi(\kappa_i+1)\partial[\nu_y]/\partial t$ is defined as the dislocation density in region *i*, $[\nu_y]$ is the crack-opening displacement, $\kappa = 3-4\nu$ and $\eta = \nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ and $\eta = 0$ for plane stress, and the $K_{ij}$ are combinations of regular and generalized Cauchy kernels that can be recovered from Ref. [1]. The zero-traction condition along the crack line is enforced by summing to zero the stress contributions from Eq. (1) and (2).

The asymptotic behavior of $b_i(t)$ was studied in detail in Ref. [1], where the loading was associated with a far-field stress consistent with a nominal stress intensity factor, rather than with an expanding inhomogeneity. Note that if one is interested in calculating the stress intensity factor produced by a far-field loading interacting with the eigenstrain within the inhomogeneity, then an appropriate superposition procedure must be performed. The dislocation densities $b_i(t)$ can be expressed as follows:

$$b_1(t) = \frac{g_1(t)}{(t-w)^{0.5-\mu}(t-a)^\mu} \tag{3a}$$

$$b_2(t) = \frac{g_2(t)}{(t-w)^{0.5}(a-t)^\mu} \tag{3b}$$

where the dominant singularity at the interface, $\mu$, and the unknown regular functions $g_i(t)$ satisfy

$$(1-\beta^2)(1+\cos^2\mu\pi) + 2[2\alpha\beta - 1 - (2\alpha\beta-\beta^2)\cos\mu\pi] + 4\mu(2-\mu)[(\alpha-\beta)^2(1-\mu)^2 - \alpha\beta + \beta(\alpha-\beta)\cos\mu\pi] = 0 \tag{4a}$$

$$g_1(t) = 0 \quad \text{at} \quad t = \infty \tag{4b}$$

$$g_2(a)/g_1(a) = \frac{(1+\alpha)\beta + (\alpha-\beta)(1-\beta)(-1+4\mu-2\mu^2) - (1-\beta^2)\cos(\mu\pi)}{(1+\alpha)(-1+2\beta-2\beta\mu)} \times \frac{\mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1)} \times (a-w)^\mu. \tag{4c}$$

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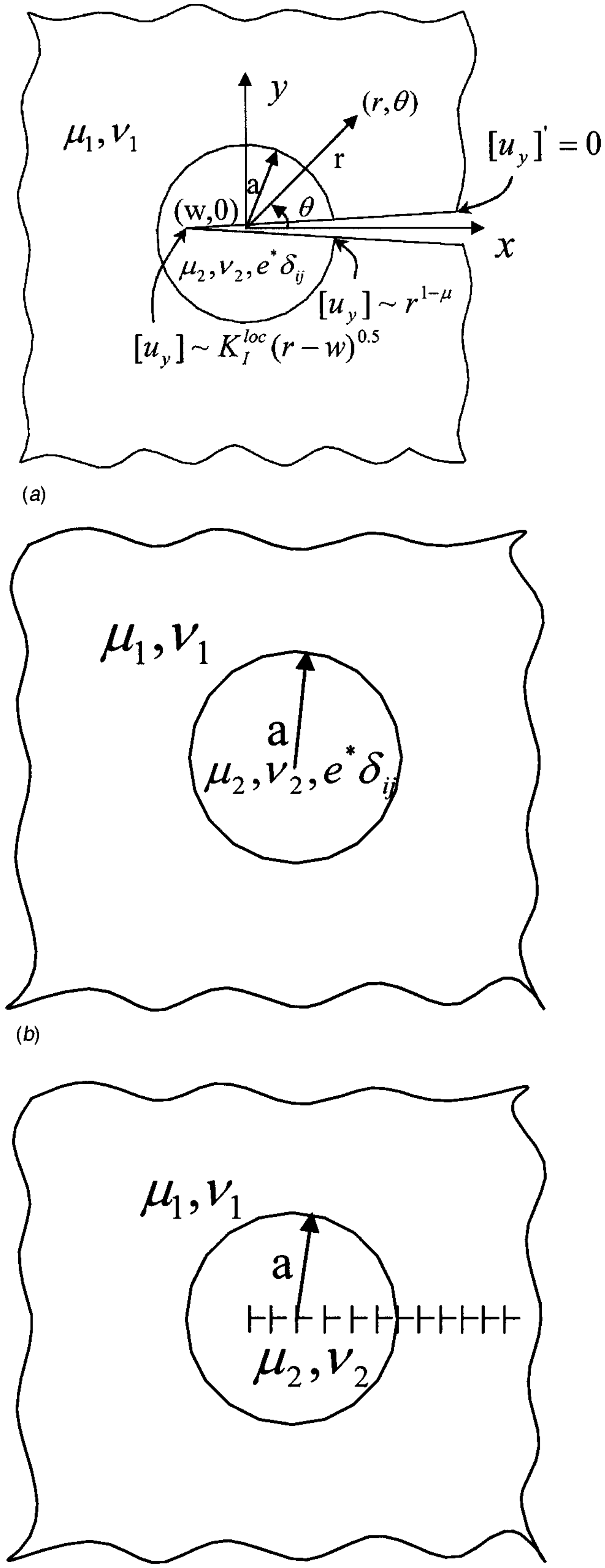


Fig. 1 (***a***) **A semi-infinite crack penetrating a circular inhomogeneity which is experiencing an isotropic eigenstrain; (*b*) an uncracked infinite plane containing a transforming circular inhomogeneity; (*c*) an infinite plane containing a continuous distribution of edge dislocations**

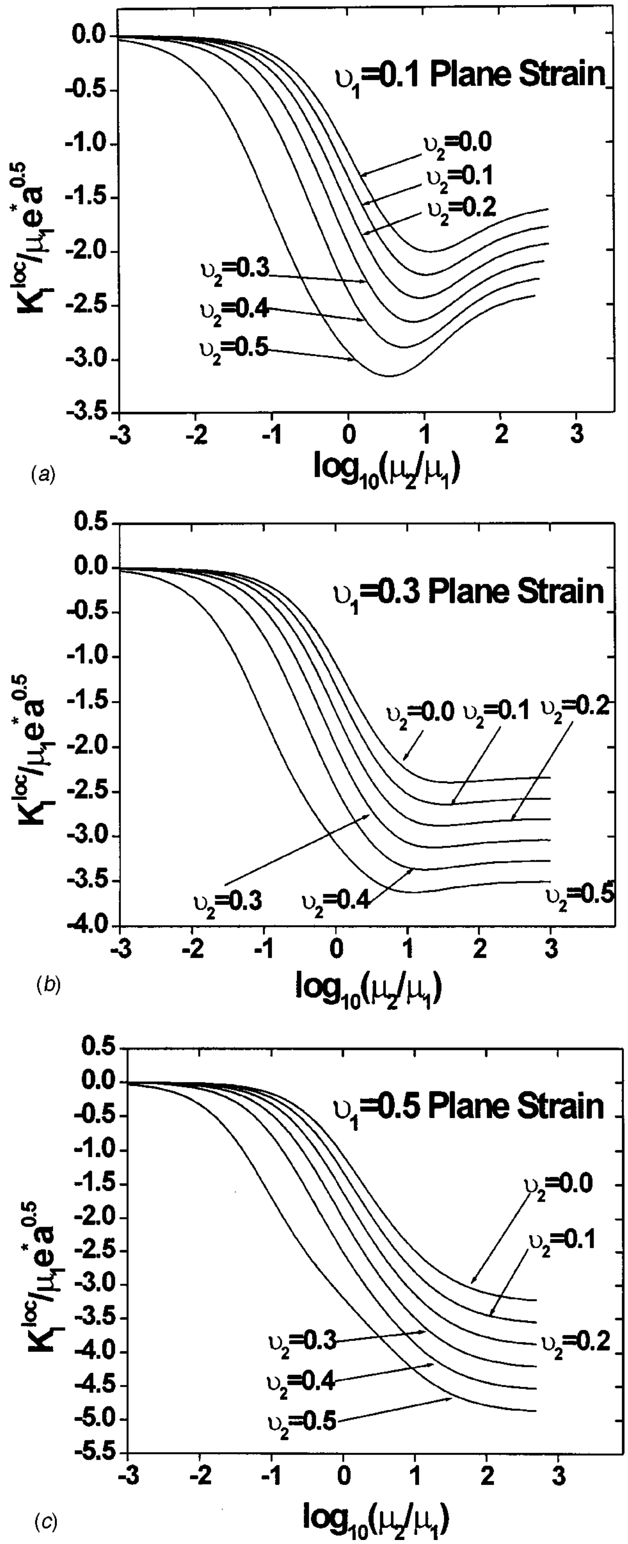


**Fig. 2 Plane strain nondimensional stress intensity factor as functions of shear modulus ratio, $\mu_2/\mu_1$, for several combinations of Poisson's ratios $\nu_1, \nu_2$**

Using the numerical approach developed by Erdogan et al. [2], which relies on the properties of Jocobi polynomials, the values of $g_i(t)$ are calculated at discrete points and the stress intensity factor is recovered as $K_I^{\mathrm{loc}} = 2\pi\sqrt{2\pi}/(a-w)^{\mu} g_2(w)$. It should be noted that because the integral equations are not homogeneous, no stabilization procedure is required to calculate a unique solution.

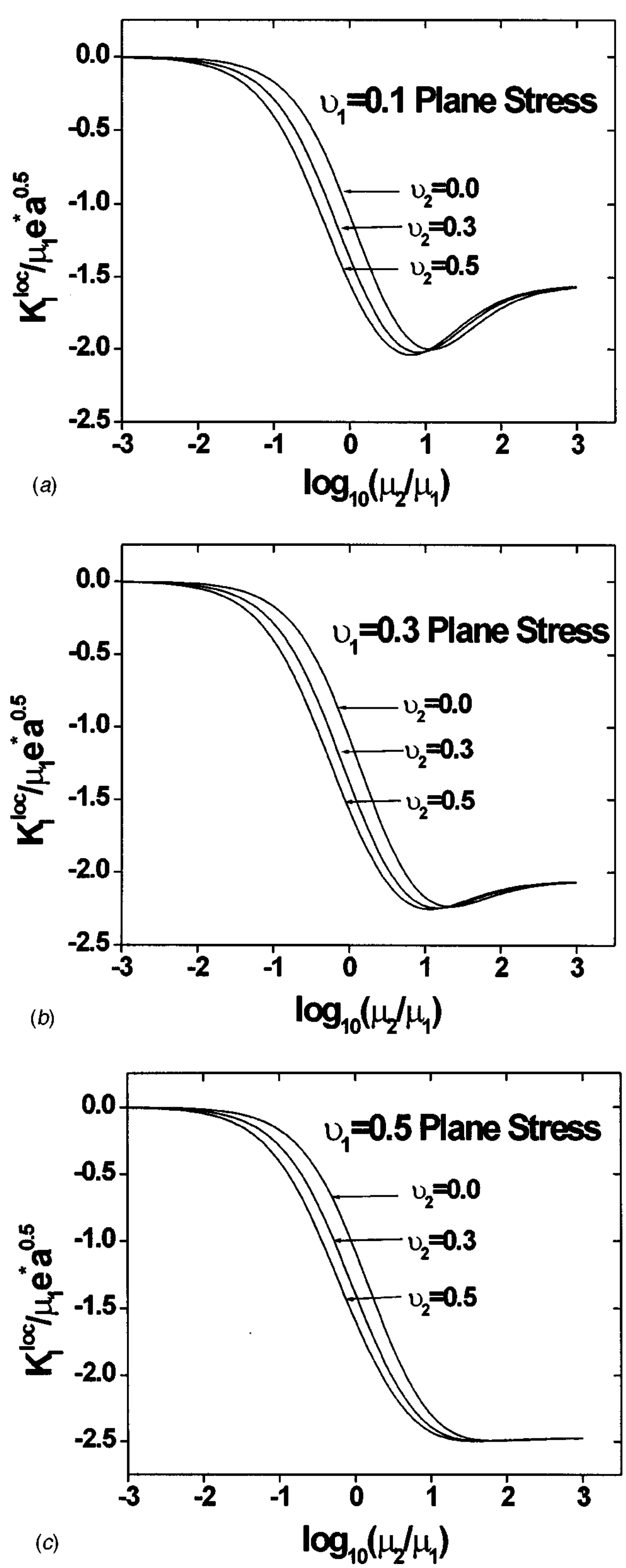


**Fig. 3 Plane stress nondimensional stress intensity factor as functions of shear modulus ratio, $\mu_2/\mu_1$, for several combinations of Poisson's ratios $\nu_1,\nu_2$**

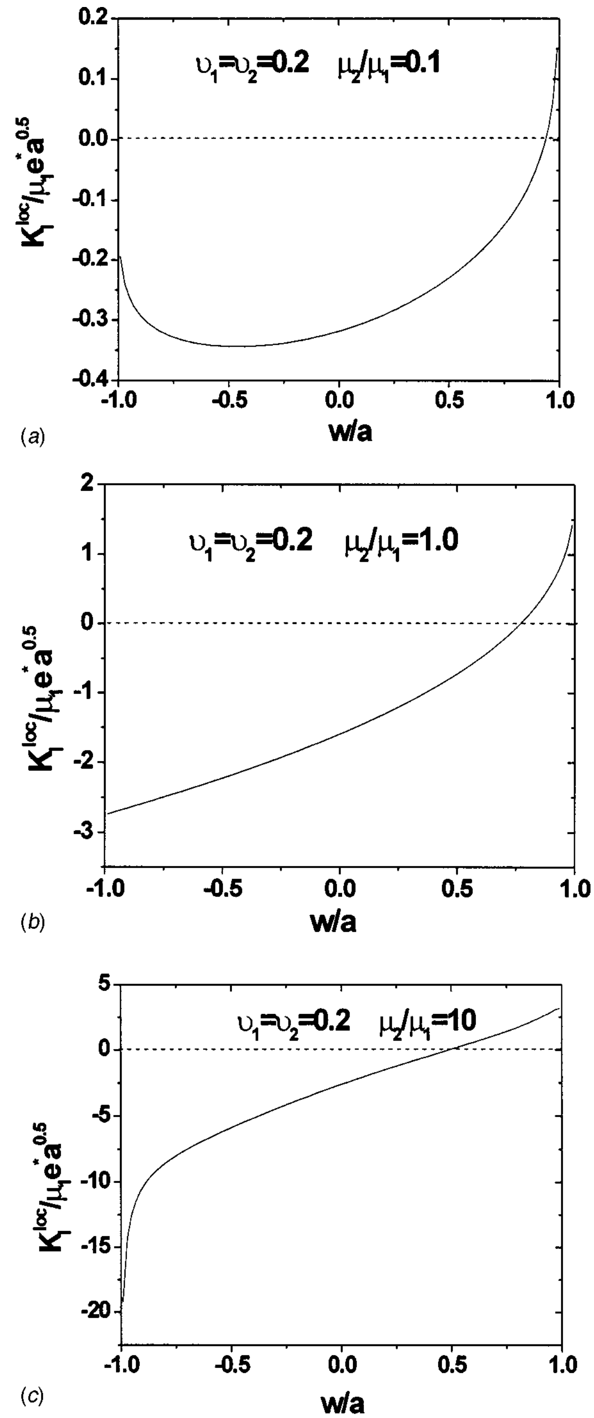


**Fig. 4 Variation of the plane strain nondimensional stress intensity factor with crack-tip position, for several combinations of elastic mismatch**

However, if the loading is associated with a far-field stress intensity factor rather than an eigenstrain within the inhomogeneity, [1], then the integral equations become homogeneous, and a stabilization procedure is required for a unique solution, [3].

## Results

The nondimensional stress intensity factor is defined as

$$\frac{K_I^{\mathrm{loc}}}{\mu_1 e^* \sqrt{a}} = \frac{2\pi\sqrt{2\pi} g_2(w)}{(a-w)^{\mu}\mu_1 e^* \sqrt{a}} = h\left(\frac{\mu_2}{\mu_1}, \nu_1, \nu_2, \frac{a-w}{a}\right). \quad (5)$$

For the crack tip at the center of the inhomogeneity $(w=0)$, $h(\mu_2/\mu_1)$ for various Poisson's ratios is presented in Figs. 2(*a–c*) for plane strain and Figs. 3(*a–c*) for plane stress. For relatively small levels of material mismatch, the number of integration points required to achieve converged stress intensity factors is approximately 20. However, large levels require a significantly higher number of integration points; the converged results presented in this note were obtained using 300 points. For positive $e^*$, the stress intensity factor is always negative, indicating crack-tip shielding, and shows a very strong dependence on elastic mismatch, the dependence being greater for plane strain than for plane stress. As expected, $K_l^{\mathrm{loc}}$ approaches zero as the inhomogeneity becomes much more compliant than the matrix, and approaches a constant value indicated by dashed lines as the inhomogeneity becomes rigid.

An interesting result of this analysis is that the crack tip is not always shielded. As shown in Fig. 4(*a*–*c*), a positive stress intensity factor, indicating amplification, results for crack tips that have entered but have not reached the center of the inhomogeneity.

The results presented above generalize those calculated in Ref. [4], where a transformation toughening model is developed for an elastically homogeneous plate. For this case, the plane-strain normalized stress intensity factor reduces to

$$\frac{K_l^{\mathrm{loc}}}{\mu_1 e^* \sqrt{a}} = -\frac{16}{3\sqrt{8\pi}}\left(\frac{1+\nu}{1-\nu}\right). \tag{6}$$

The results presented in Fig. 2 corresponding to uniform elastic moduli match Eq. (6) to within three significant figures.

## References

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