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COMPLIANCE MATRICES FOR CRACKED BODIES

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Two dimensional problems in linear elastic fracture mechanics are often reduced to a singular integral equation (or system of integral equations) of the form

$$\int_{-1}^{1} \frac{f(t)\,dt}{t-x} + \int_{-1}^{1} K(x,t)\, f(t)\, dt = g(x), \quad -1<x<1 \tag{1}$$

where f(t) is a function to be determined, and K(x,t) and g(x) are known functions related to the geometry of the cracked body and the loading on the crack surface(s), respectively. The function f(t), which is called the dislocation density, is the slope of the crack profile and, if the crack is closed at its tips, satisfies the consistency condition

$$\int_{-1}^{1} f(t)\, dt = 0 \tag{2}$$

In this paper an algorithm is presented which can be used to develop a compliance matrix for a cracked solid when such a formulation is used. This matrix relates the stresses at the collocation points arising from the applied loads (including those applied to the crack surfaces) to the values of the opening of the crack at the integration points. The method relies on the numerical procedure developed by Gerasoulis [1], which is used to reduce (1) to a system of algebraic equations for unknown values of f(t) at discrete points in the interval [-1,1].

The algorithm is best explained through a simple example. For the cracked plate shown in Fig. 1 the governing equations for the dislocation density are

$$\frac{2\mu}{\pi(\kappa+1)} \int_{-1}^{1} \frac{f(t)\,dt}{t-x} = \sigma_\infty, \quad -1<x<1 \tag{3}$$

$$\int_{-1}^{1} f(t)\, dt = 0 \tag{4}$$

where μ is the shear modulus, $\kappa = 3-4\nu$ for plane strain, and ν is Poisson's ratio.

The crack opening displacement is given by

$$u^+ - u^- = \int_{x}^{1} f(t)\, dt \tag{5}$$

The exact solution to this problem is

$$f(x) = [(\kappa+1)\sigma_\infty/2\mu]\ [x/\sqrt{(1-x^2)}] \tag{6}$$

$$u^+ - u^- = [(\kappa+1)\sigma_\infty/2\mu]\ /\ \sqrt{(1-x^2)} \tag{7}$$

Following [1] (and nondimensionalizing by setting $2\mu/\pi\sigma_\infty(\kappa+1)$ equal to unity), $f(t)$ is expressed by

$$f(t) = \phi(t)/\sqrt{(1-t^2)} \tag{8}$$

and $\phi(t)$ is approximated as piecewise quadratic in [-1,1]. The result is that (3) and (4) are reduced to a system of algebraic equations through quadrature formulas. The details of the quadrature can be found in [1] and are omitted here. The results are

$$\sum_{i=1}^{2N+1} w_i(x_k)\quad \phi(t_i) = 1 \quad k=1,2N \tag{9}$$

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$$\sum_{i=1}^{2n+1} v_i\ \phi(t_i) = 0 \tag{10}$$

We note that the unknown values of ϕ are those at the integration points t_i, while the stresses on the right hand side of (9) are at the collocation points x_k.

For illustration we take five points for the quadrature, and (9) and (10) become

$$\begin{bmatrix} -3.236 & 1.527 & .699 & .683 & .328 \\ -.948 & -2.311 & 1.626 & 1.190 & .443 \\ -.443 & -1.190 & -1.626 & 2.311 & .948 \\ -.328 & -.683 & -.699 & -1.527 & 3.236 \\ .571 & .858 & .283 & .858 & .571 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

$$G_{ik} \quad \mathrm{x} \quad \phi_i = L_k$$

In (11), matrix G is characteristic of the geometry of the problem and matrix L represents the loading and crack closure condition. What is usually of interest in such a problem is the stress intensity factor, and since the stress intensity factor is proportional to the value of ϕ at the endpoints [1], (11) is solved for the unknown vector ϕ.

Instead of doing this, in this report the inverse of matrix G is obtained, and the product of this matrix and $1/\sqrt{(1-t^2)}$ is integrated term by term to obtain a matrix C which is called the compliance matrix for this particular geometry. The integration of each term is performed using the weights for the Lagrange interpolation polynomials, since the function ϕ is approximated in this manner. The results of the integration lead to the following

$$\begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.087 & 0.121 & 0.049 & 0.019 \\ 0.029 & 0.130 & 0.130 & 0.029 \\ 0.019 & 0.049 & 0.121 & 0.087 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (u^+ - u^-)_1 \\ (u^+ - u^-)_2 \\ (u^+ - u^-)_3 \\ (u^+ - u^-)_4 \\ (u^+ - u^-)_5 \end{bmatrix} \tag{12}$$

$$C_{ik} \quad x \quad \sigma_{x_k} = (u^+ - u^-)_{t_i}$$

We note that matrix C is not a square matrix. This is because the number of integration points is one more than the number of collocation points. Matrix C_{ik} relates the displacement of the crack faces at the point t_i to the stress at point x_k.

For the present problem the applied loading vector is unity, and premultiplying it by the compliance matrix leads to

$$\begin{aligned} (u^+ - U^-)_1 &= 0.0 \\ (u^+ - U^-)_2 &= 0.276 \\ (u^+ - u^-)_3 &= 0.318 \\ (u^+ - u^-)_4 &= 0.276 \\ (u^+ - u^-)_5 &= 0.0 \end{aligned} \tag{13}$$

The above displacements agree with the exact solution.

The utility of the compliance matrix becomes evident when one is interested in investigating the effects of many loading cases, and in particular, if the stresses along the crack surfaces depend on the crack opening displacements. Such loadings are used in models for fiber reinforced concrete, rocks, ceramics, and other materials where microcracking, fiber bridging, and other nonlinear effects are modeled as nonlinear springs along the crack surfaces [2,3,4,5]. For these models (1) becomes nonlinear, and an iterative solution is needed. With the use of the compliance matrix, this iteration procedure is efficient and rapid.

As an example, let us assume that the crack surfaces are bridged by fibers, and that the stresses transmitted by the fibers to the crack depend on the opening of the crack. The displacements along the crack will be governed by

$$(u^+ - u^-)_{t_i} = C_{ik} \times [\sigma_{x_k}(\text{applied loads}) + \sigma_{x_k}(u^+ - u^-)] \tag{14}$$

where now the stresses are decomposed into those arising from the applied loads, and those due to the fiber bridging. The function $\sigma(u^+ - u^-)$ is determined from experiments [6]. For the first iteration the stresses due to fiber bridging are assumed to be equal to zero. Premultiplying the stresses arising from the applied loading by the compliance matrix results in the first approximation to the crack opening displacements. From these displacements the first approximation to the fiber bridging stresses are determined, and these are applied to the crack surfaces. The procedure is repeated until convergence is reached. This procedure was used in [4] (where experiments performed on concrete and fiber reinforced concrete were analyzed) and convergence was observed to be very fast (only several iterations were needed for three and four point bending specimens).

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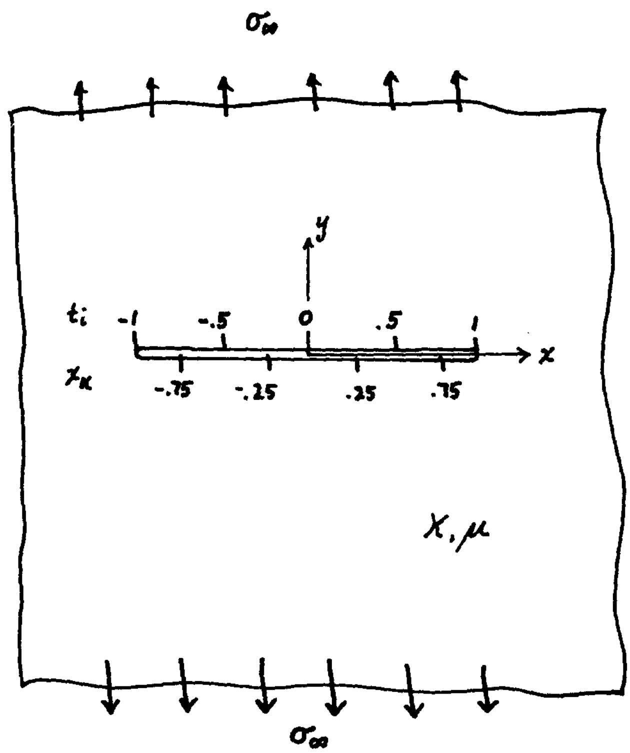

Figure 1. Cracked plate showing integration and collocation points for five point quadrature