A Crack Very Close to a Bimaterial Interface

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This paper presents the plane elastostatics analysis of a semi-infinite crack perpendicular to a perfectly bonded bimaterial interface. Both cases of the crack approaching the interface and penetrating the interface are addressed. The distance from the tip of the crack to the interface is δ. A singular integral equation approach is used to calculate the stress intensity factor, K_I*, and the crack-opening displacement at the interface,* η*, as functions of δ, the Dundurs parameters α and β, and the stress intensity factor* k_I *associated with the same crack terminating at the interface (the case* $\delta = 0$*). The results are presented as* $K_I = k_I \delta^{1/2-\lambda} f(\alpha, \beta)$ *and* $\eta = C k_I \delta^{1-\lambda} \tilde{\eta}(\alpha, \beta)$ *where λ is the strength of the stress singularity associated with* $\delta = 0$*, f and* $\tilde{\eta}$ *are functions calculated numerically and C is a material constant. These results can be used to determine the stress intensity factor and crack opening displacement of cracks of finite length 2a with one tip at a distance δ from the interface for* $\delta/a \ll 1$*. The selected results presented for a crack loaded by a uniform far-field tension in each half-plane show that the stress intensity factors approach their limits at a relatively slow rate.*

1 Introduction

Consider the plane elastostatics problem shown in Fig. 1(*a*). A Mode I crack of length $2a$ is perpendicular to the perfectly bonded interface between two isotropic half-planes with shear moduli μ_i and Poisson's ratios ν_i, $i = 1, 2$. The distance from the left tip of the crack to the interface is δ. Because of its relevance to fracture of composite materials, the problem of calculating the stress intensity factors for this configuration has been addressed by several authors (Erdogan et al., 1973; Atkinson, 1975). It is well known that this elasticity problem can be formulated using the Green's function for the stress produced along the crack line by an edge dislocation. This procedure leads to the following singular integral equation and uniqueness condition:

$$\frac{2\mu_1}{\pi(\kappa_1+1)} \int_\delta^{\delta+2a} b(\xi)\left[\frac{1}{y-\xi} + \frac{\alpha+\beta^2}{1-\beta^2}\frac{1}{y+\xi} + \frac{2(\alpha-\beta)}{1+\beta}\frac{\xi(y-\xi)}{(y+\xi)^3}\right]d\xi = -\sigma_{xx}(y)$$

$$\int_\delta^{\delta+2a} b(\xi)d\xi = 0 \quad \delta \le y, \xi \le \delta + 2a \qquad (1)$$

where σ_{xx} is the stress along the crack line induced by the remote loading in the uncracked body, α and β are the Dundurs parameters (Dundurs, 1969)

$$\alpha = \frac{\mu_2(\kappa_1+1) - \mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)};$$

$$\beta = \frac{\mu_2(\kappa_1-1) - \mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)}$$

$\kappa_i = 3 - 4\nu_i$ for plane strain and $\kappa_i = (3 - \nu_i)/(1 + \nu_i)$ for plane stress. The unknown dislocation density $b(\xi)$ is defined in terms of the crack opening displacement $[u_x(\xi)]$ as

$$b(\xi) \equiv -\frac{\partial}{\partial \xi}[u_x]. \qquad (2)$$

The first equation in (1) represents the zero traction condition along the crack surfaces, while the second enforces single-valued displacements (both crack tips are closed). In the following, the loading is taken as uniform remote tension in each half-plane, $\sigma^{(1)}$ and $\sigma^{(2)}$, such that

$$\sigma^{(2)} = \frac{1+\alpha}{1-\alpha}\sigma^{(1)} \qquad (3)$$

and therefore $\sigma_{xx} = \sigma^{(1)}$ in Eq. (1).

The ratio $\delta/2a$ enters in the kernel of the singular integral Eq. (1) in such a way that stress intensity factor values calculated using a direct numerical solution inevitably lose accuracy for $\delta/2a \ll 1$. Indeed, the smallest ratio for which Erdogan et al. (1973) present results is $\delta/2a = 0.05$. Their results showed that as $\delta \to 0$ the stress intensity factor of the crack tip closest to the interface approaches zero when $\mu_2 > \mu_1$ and infinity when $\mu_1 > \mu_2$. These limits result from the discontinuous change in the order of the stress singularity as δ becomes equal to zero. As will be explained in the next section, for $\delta = 0$ and $\mu_2 > \mu_1$ the stress ahead of the crack tip is of the order $k_I r^{-\lambda}$ with $\lambda < \frac{1}{2}$. This weaker singularity in effect reduces to zero, as $\delta \to 0$, the amplitude K_I of the square root singularity associated with $\delta \neq 0$. For $\mu_1 > \mu_2$, $\lambda > \frac{1}{2}$, and similar arguments explain why K_I increases to infinity as $\delta \to 0$. Assuming linear elastic fracture mechanics these limits imply that the crack reaches the interface at infinite load for $\mu_2 > \mu_1$ and zero load for $\mu_1 > \mu_2$.

As a first step toward the development of physically sound propagation criteria for interface cracks, this paper is concerned with determining, as functions of the elastic mismatch, the rate at which the square root singularity approaches the limits discussed above. To this end the problem is formulated asymptotically in terms of a semi-infinite crack in which the only length parameter is δ.

The approach used is essentially the same as that used by Hutchinson et al. (1987) to study a crack very close to and parallel to a bimaterial interface. It relies on some relevant well-

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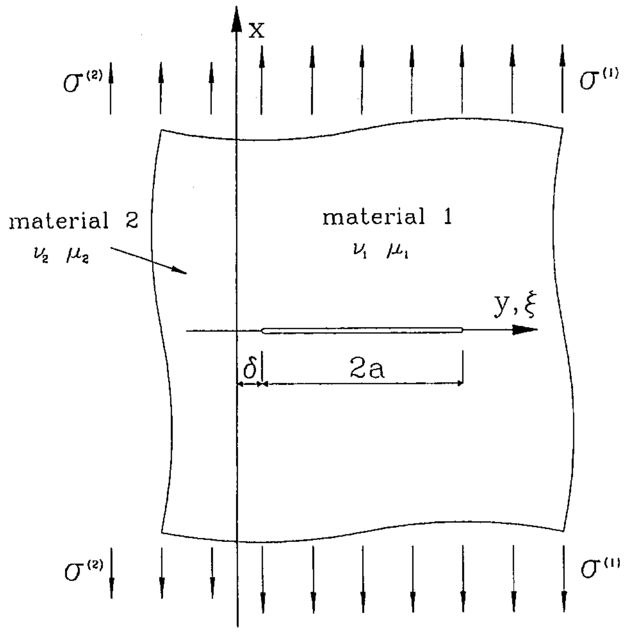

Fig. 1(*a*)

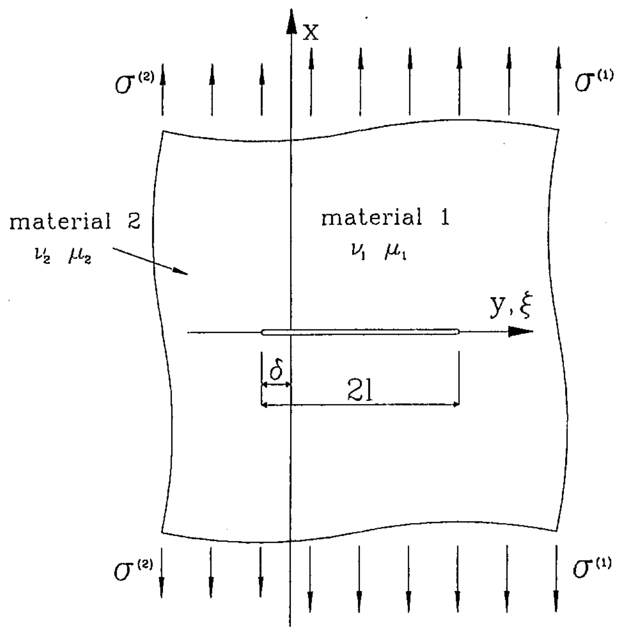

Fig. 1(*b*)

Fig. 1 Finite length crack (*a*) approaching and (*b*) penetrating a bimaterial interface

known results for the asymptotic behavior of the stresses and crack-opening displacements in the vicinity of the tip of a crack impinging on a bimaterial interface. These are reviewed briefly in the next section. The solution of the semi-infinite crack problem is discussed in the third section and an example of how the asymptotic analysis can be applied to the problem of Fig. 1(*a*) is provided in the 4th section. The last section extends the method to the case of a finite crack that penetrates a distance δ through the interface, Fig. 1(*b*).

2 Finite Crack Terminating at the Interface

The results of the Williams technique analysis for a crack terminating at the interface ($\delta = 0$ in Fig. 1(*a*)) show that the traction ahead of the crack tip is characterized by

$$\lim_{y\to 0^-} \{\sigma_{xx}^{(2)}(y)\} = \frac{k_I}{\sqrt{2\pi}}(-y)^{-\lambda} \tag{4}$$

where superscript (i) denotes "in material (i)," k_I is the stress intensity factor, and $\lambda (0 \le \lambda < 1)$ is the root of the equation derived by Zak and Williams (1963)

$$\cos(\lambda\pi) = \frac{2(\beta-\alpha)}{(1+\beta)}(1-\lambda)^2 + \frac{\alpha+\beta^2}{1-\beta^2}. \tag{5}$$

The loci of constant λ in the α-β plane are shown in Fig. 2(*a*). As pointed out by Dundurs (1970), this figure clearly illustrates that for $\alpha \to 1$ the quantity λ is more sensitive to the mismatch in the Poisson's ratios, while for $\alpha \to -1$ it is more sensitive to the mismatch in the shear moduli.

Another important result from the Williams analysis relates the crack-opening displacement (COD), $[u_x^{(1)}(y)]$, to the stress ahead of the crack, i.e.,

$$\lim_{y\to 0^+} \{\sigma_{xx}^{(2)}(-y)\} = \lim_{y\to 0^+} \left\{-\psi \frac{\partial}{\partial y}[u_x^{(1)}(y)]\right\}$$

$$= \lim_{y\to 0^+} \{\psi b(y)\} \tag{6}$$

where the bimaterial parameter ψ is defined as

$$\psi \equiv \frac{2\mu_1}{(\kappa_1+1)} \frac{1}{\sin(\lambda\pi)} \left(\frac{1+\alpha}{1-\beta^2}\right)[1 - 2\beta(\lambda-1)]. \tag{7}$$

Equation (6) allows the determination of the stress intensity factor of a crack of length $2a$ in terms of the dislocation density as

$$k_I = \lim_{y\to 0^+} \{\psi\sqrt{\pi} a^\lambda \tilde{b}(y)\} \tag{8}$$

in which

$$\tilde{b}(y) = a^{-(\lambda+1/2)} b(y) y^\lambda (2a-y)^{1/2}$$

is the regular part of the dislocation density $b(y)$.

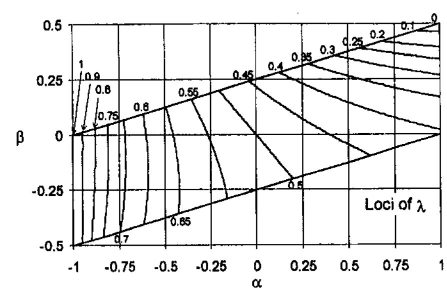

Fig. 2(*a*)

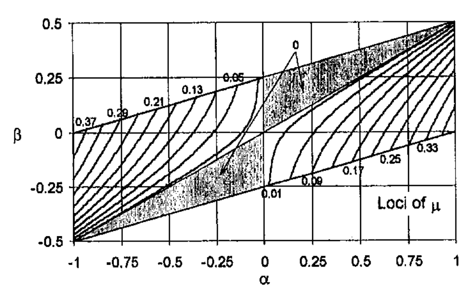

Fig. 2(*b*)

Fig. 2 Loci of (*a*) constant λ and (*b*) constant μ in the α-β plane

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It is worth noting that (8) could also be derived by studying the behavior near the crack tip of the generalized Cauchy-type integral that represents the stress $\sigma_{xx}^{(2)}$.

3 Semi-infinite Crack Analysis

Consider a semi-infinite crack perpendicular to the interface and terminating at a distance δ from it. The COD at a point $r = y - \delta$ very close to the crack tip ($r/\delta \ll 1$) is given by $[u_x] \propto K_I r^{1/2}$, where K_I is the stress intensity factor. For $r/\delta \gg 1$ the COD approaches the one associated with the crack tip impinging on the interface ($\delta = 0$), i.e., $[u_x] \propto k_I r^{1-\lambda}$. The physical meaning is that since δ is very small the COD in the far-field is indistinguishable from the COD of the same crack impinging on the interface.

Linearity and dimensional considerations (δ is the only characteristic length) demand that

$$\frac{K_I}{k_I \delta^{1/2-\lambda}} = f(\alpha, \beta) \tag{9}$$

where f is a function of the Dundurs parameters only. This type of argument was employed by Hutchinson et al. (1987) and He and Hutchinson (1989).

It should be noted that this last result was derived by Atkinson (1975) by applying the Mellin transform to the integral equation and using the Wiener-Hopf technique. He showed that the stresses ahead of the crack are given by (using the notation in his paper)

$$\sigma_{xx} \approx \epsilon^{1/2} r^{-1/2} \{ \sum_{k=1}^{N} A_k \epsilon^{s_k^r - 2} + O(\epsilon^{3s_1^r - 4}) \} \tag{10}$$

where the A_k are constants independent of ϵ, α_1 and β are material constants (not to be confused with the Dundurs parameters), and s_k^r are the N real roots ($\text{Re}(s) > 1$) of the equation

$$\cos(\pi s) + \alpha_1 - \beta(s-1)^2 = 0. \tag{11}$$

It can be easily shown that the dominant term of the stress given by (10) corresponds to that produced by the stress intensity factor defined by relation (9), the constants A_k being identified with the values of $k_I f(\alpha, \beta)$, and $s_1^r = 2 - \lambda$. Atkinson developed his solution for a constant pressure loading, but did not present numerical results for coefficients A_k. The main contribution of the present paper is that it presents complete results for these universal functions.

As will be described in the next section, (9) provides a powerful tool for the asymptotic analysis of finite length cracks approaching a bimaterial interface. The values of the function $f(\alpha, \beta)$ were calculated by integrating numerically (1) for $2a = \infty$. The details of the solution procedure are given in the Appendix.

The loci of constant f in the α-β plane are shown in Fig. 3(a). It is interesting to note that the sensitivity of f to changes in shear moduli and Poisson ratios is qualitatively the same as that of the singularity coefficient λ.

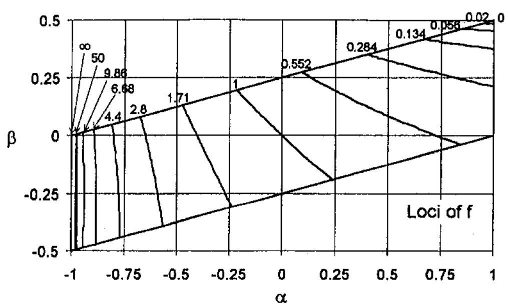

Fig. 3(*a*)

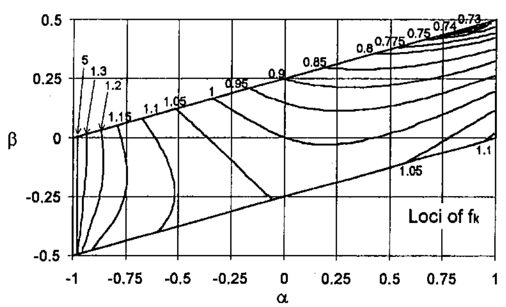

Fig. 3(*b*)

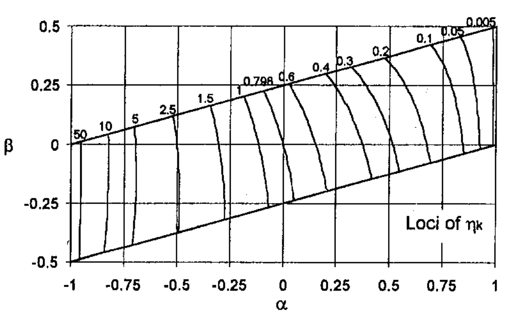

Fig. 3(c)

Fig. 3 Loci of (*a*) constant *f*, (*b*) constant *f, and (*c*) constant $\tilde{\eta}$ in the α-β plane**

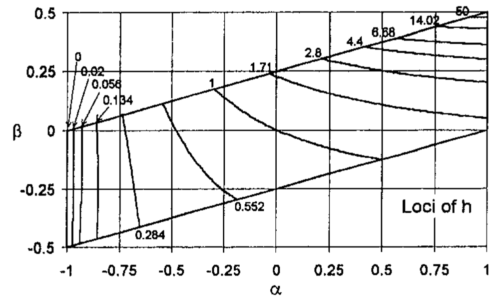

Fig. 4 Loci of constant *h* in the α, β-plane

4 Finite Crack Very Close to the Interface

The numerical scheme used for solving the singular integral equation for the finite crack depicted in Fig. 1(a) becomes unstable when the ratio $\delta/2a$ assumes very small values. For these cases, an indirect method based on asymptotical analysis is recommended for computing the stress intensity factor. This approach relies on the combination of (9) and the stress intensity factor k_I associated with the finite crack terminating at the interface ($\delta = 0$).

As an example, consider the case of a crack acted upon by a uniform remote tension field in the two connected half-planes according to (3). As shown in the Appendix the stress intensity factor k_I can be represented as

$$\frac{k_I}{\sigma\sqrt{\pi} a^{\lambda}} = h(\alpha, \beta). \tag{12}$$

The loci of constant h in the α-β plane are shown in Fig. 4. It is observed that the stress intensity factor for this problem is yet another parameter that is more sensitive to the mismatch in the Poisson's ratios for $\alpha \to 1$, while for $\alpha \to -1$ it is more sensitive to the mismatch in the shear moduli. Combining (9) and (12) leads to the following expression for the stress intensity factor of a crack of length $2a$ at a distance δ from the interface:

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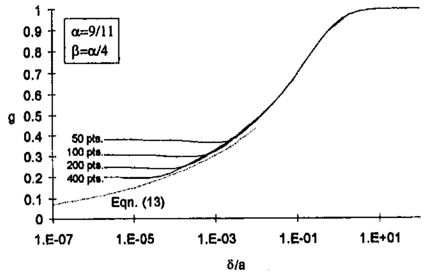

Fig. 5(*a*)

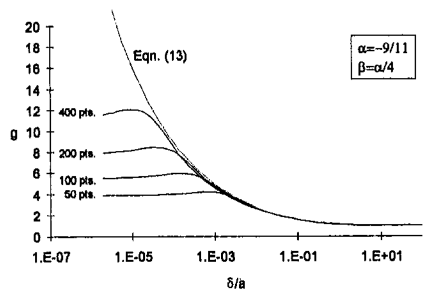

Fig. 5(*b*)

Fig. 5 Numerical instability and transition to the asymptotical solution of g for small δ/a ratios for (*a*) $\alpha = 4\beta = \frac{9}{11}$ and (*b*) $\alpha = 4\beta = -\frac{9}{11}$

$$\frac{K_I}{\sigma\sqrt{\pi}a^{1/2}} = g(\alpha, \beta) = \left(\frac{\delta}{a}\right)^{1/2-\lambda} f(\alpha, \beta)h(\alpha, \beta); \quad \delta/a \to 0. \tag{13}$$

Selected results for the asymptotic value of the stress intensity factor as calculated using (13) are presented in Figs. 5(*a*) and 5(*b*) for $\alpha = 4\beta = \pm\frac{9}{11}$; these values of the Dundurs parameters include $\nu_1 = \nu_2$ and $\mu_2/\mu_1 = 10$ or $\mu_1/\mu_2 = 10$. The solid lines in these plots are the values of the stress intensity factor calculated through a direct numerical solution of (1) along the finite interval. The procedure used for these calculations is outlined in the Appendix. As expected the direct solution for a given number of integration points breaks down as the distance from the crack tip to the interface assumes relatively small values. The asymptotic solution approaches the envelope defined by the value of δ/a at which the direct numerical solution, for a given value of integration points, becomes unstable.

The most interesting results of this analysis is that the stress intensity factor approaches the aforementioned limits at relatively slow rates. For significant elastic mismatch, $\alpha = 4\beta = \frac{9}{11}$, the stress intensity factor for $\delta/a = 0.001$ is approximately 30 percent of the nominal value associated with no interface. These results suggest that although the stress intensity factor for $\mu_2 > \mu_1$ approaches zero, this limit is associated with distances δ much smaller than the plastic zone that inevitably surrounds the crack tip. The leading edge of the plastic zone will thus reach the interface at a finite load. Perhaps more importantly, cracks in typical engineering materials will have extremely small δ values that will invalidate a continuum mechanics formulation.

5 Finite Crack Extending Through the Interface

The asymptotical technique described in the previous sections can be easily extended to other interface crack problems. The natural extension of the previous formulation is to a finite crack of length $2l = 2(a + b)$ that has extended beyond the interface a distance $\delta = 2b \ll l$ (Fig. 1(*b*)). This problem can be reduced to a set of coupled singular integral equations using the same Green's function approach that is used to derive (1) (Erdogan and Biricikoglu, 1973). These equations are written symbolically in terms of the dislocation densities $b^{(i)}(\xi)$ $(i = 1, 2)$:

$$A_1 \int_0^{2a} b^{(1)}(\xi)K_{1i}\,d\xi + A_2 \int_{-2b}^{0} b^{(2)}(\xi)K_{2i}\,d\xi = -\sigma^{(i)} \qquad (i = 1, 2)$$

$$\int_0^{2a} b^{(1)}(\xi)d\xi + \int_{-2b}^{0} b^{(2)}(\xi)d\xi = 0 \tag{14}$$

where K_{ij} $(i, j = 1, 2)$ are Cauchy-type kernels. The first two Eqs. (14) represent the traction boundary conditions, while the third enforces single-valued displacements. The condition on the dislocation density required to insure compatibility at the interface is given by

$$\lim_{y \to 0^+} b^{(2)}(y)/b^{(1)}(-y) = F(\alpha, \beta, \mu) \tag{15}$$

where

$$F(\alpha, \beta, \mu) = \frac{(1 + \alpha)\beta + (\alpha - \beta) \times (1 - \beta)(-1 + 4\mu - 2\mu^2) - (1 - \beta^2)\cos(\mu\pi)}{(1 + \alpha)(-1 + 2\beta - 2\beta\mu)} \tag{16}$$

and μ is the power of the stress singularity, which satisfies the characteristic equation

$$(1 - \beta^2)(1 + \cos^2 \mu\pi) + 2[2\alpha\beta - 1 - (2\alpha\beta - \beta^2)\cos \mu\pi] + 4\mu(2 - \mu) [(\alpha - \beta)^2(1 - \mu)^2 - \alpha\beta + \beta(\alpha - \beta)\cos \mu\pi] = 0.$$

The loci of constant μ are plotted in Fig. 2(*b*).

Again a direct solution of Eqs. (14) is inadequate for very small b/a ratios and an alternative approach is furnished by the asymptotical analysis. The reference problem is still the finite crack of length $2l$ terminating at the interface. The semi-infinite analysis on the other hand has to be redefined, since the crack tip is now located beyond the interface at a distance δ from it. As before, the far-field COD has to approach the one associated with no penetration, $[u_x] \propto k_I r^{1-\lambda}$, and in the vicinity of the crack tip the COD is given by $[u_x] \propto K_I r^{1/2}$. However, an additional requirement is that the COD at the interface be of the order $r^{1-\mu}$. Equation (9) still applies with $f(\alpha, \beta)$ replaced by the new function $f^*(\alpha, \beta)$ whose values are computed by solving numerically the proper set of integral equations (see Appendix); the loci of this function in the α-β plane are plotted in Fig. 3(*b*).

Combining (9) and (12) leads to the asymptotical expression of the stress intensity factor of a finite crack of length $2l$ the has penetrated in material 2 by the distance $\delta = 2b \ll l$

$$\frac{K_I}{\sigma^{(1)}\sqrt{\pi}l^{1/2}} = g^*(\alpha, \beta) = \left(\frac{2c}{1 + c}\right)^{1/2-\lambda} f^*(\alpha, \beta)h(\alpha, \beta) \tag{17}$$

where

$$c = b/a.$$

The predictions of Eq. (17) are valid in the limit $\delta/l \to 0$. Figures 6 and 7 show the convergence of the nondimensional stress intensity factor values found by direct numerical integration to the asymptotic solution given by (17) for the two material combinations already used in the previous sections, $\alpha = \pm\frac{9}{11} = 4\beta$. Note that the for the case $\alpha = +\frac{9}{11} = 4\beta$, which corresponds to $\lambda = 0.347$, the rate of convergence is much slower than for $\alpha = -\frac{9}{11} = 4\beta$ $(\lambda = 0.755)$.

Note that the asymptotic analysis can be used to compute not only stress intensity factors but also other quantities such as the

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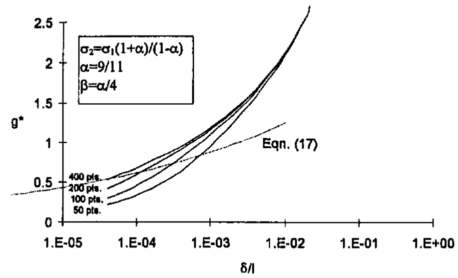

Fig. 6 Numerical instability and transition to the asymptotical solution of g^* for small δ/a ratios for $\alpha = 4\beta = \frac{9}{11}$

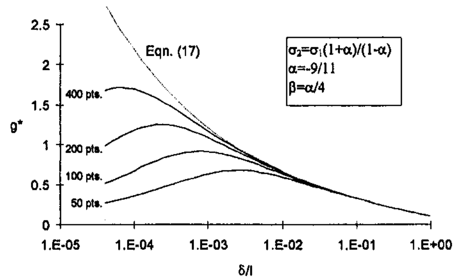

Fig. 7 Numerical instability and transition to the asymptotical solution of g^* for small δ/a ratios for $\alpha = 4\beta = -\frac{9}{11}$

crack-opening displacement at the interface. In fact, for a semi-infinite crack the COD at the interface, η, is given by

$$\eta = \frac{1 + \kappa_1}{2\mu_1} k_l \delta^{1-\lambda} \tilde{\eta}(\alpha, \beta) \tag{18}$$

where $\tilde{\eta}$ is a function of the Dundurs parameters whose loci are shown in Fig. 3(b). The asymptotic expression of the crack-opening displacement at the interface for the finite crack in Fig. 1(b) is then given by

$$\eta_f = -\sqrt{\pi}\phi_1 \delta \left(\frac{l}{\delta}\right)^{\lambda} h(\alpha, \beta)\tilde{\eta}(\alpha, \beta) \quad \delta/l \to 0 \tag{19}$$

where

$$\phi_1 = -\frac{\sigma^{(1)}(\kappa_1 + 1)}{2\mu_1}.$$

6 Conclusions

The numerical schemes that are used to solve integral equations describing the elastostatics problems of finite length cracks close to a bimaterial interface are not accurate when the relative distance δ from the crack tip to the interface becomes very small. An asymptotic analysis has been developed that provides accurate stress intensity factors for such problems and gives insight into their rate of change as $\delta \to 0$. For the case of a crack approaching or penetrating a bimaterial interface, it has been shown that the stress intensity factor at the leading crack tip approaches its limiting value at a slow rate. These results suggest that propagation criteria for such problems are associated with nonlinear processes. The technique presented in this paper can be used to solve a class of problems in which a small parameter leads to an unstable direct numerical solution of the governing equations.

Acknowledgment

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APPENDIX

Let

$$A_i = \frac{2\mu_i}{\pi(1 + \kappa_i)} \quad (i = 1, 2)$$

$$M = \frac{\alpha + \beta^2}{1 - \beta^2}; \quad N = \frac{2(\alpha - \beta)}{1 + \beta}; \quad S = \frac{1 + \alpha}{1 - \beta^2};$$

$$P = \frac{\alpha - \beta^2}{1 - \beta^2}; \quad Q = \frac{2(\alpha - \beta)}{1 - \beta}; \quad T = \frac{1 - \alpha}{1 - \beta^2};$$

$$K_{11}(\xi, y) = \frac{1}{y - \xi} + \frac{M}{y + \xi} + \frac{\xi(y - \xi)N}{(y + \xi)^3};$$

$$K_{12}(\xi, y) = S\left[\frac{1}{y - \xi} - 2\beta \frac{\xi}{(y - \xi)^2}\right];$$

$$K_{21}(\xi, y) = T\left[\frac{1}{y - \xi} + 2\beta \frac{\xi}{(y - \xi)^2}\right];$$

$$K_{22}(\xi, y) = \frac{1}{y - \xi} - \frac{P}{y + \xi} - \frac{\xi(y - \xi)Q}{(y + \xi)^3}. \tag{A1}$$

The following procedures were used to solve the integral equations numerically.

Crack of Length $2a$ at Distance δ From the Interface ($\delta \geq 0$)

The integral equations for this case are

$$A_1 \int_{\delta}^{\delta+2a} b(\xi) K_{11}(\xi, y) d\xi = -\sigma^{(1)}$$

$$\int_{\delta}^{\delta+2a} b(\xi) d\xi = 0 \quad \delta \leq y, \xi \leq \delta + 2a$$

The case $\delta = 0$ corresponds to a crack impinging on the interface. For the numerical computation these equations are rendered in nondimensional form and normalized in the interval $[-1, 1]$ by means of the transformations

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$$t = \frac{\xi - (a + \delta)}{a}; \quad \zeta = \frac{y - (a + \delta)}{a};$$

$$\phi = -\frac{\sigma^{(1)}(\kappa_1 + 1)}{2\mu_1}.$$

The nondimensional stress intensity factor at the crack tip closest to the interface is given by

$$h(\alpha, \beta) = \frac{k_I}{\sigma^{(1)}\sqrt{\pi} a^{\lambda}} = \frac{\kappa_1 + 1}{2\mu_1} \psi \tilde{b}(-1); \quad \delta = 0 \quad \text{(A3)}$$

$$g(\alpha, \beta) = \frac{K_I}{\sigma^{(1)}\sqrt{\pi} a^{1/2}} = \tilde{b}(-1); \quad \delta > 0 \quad \text{(A4)}$$

where

$$\psi \equiv \frac{2\mu_1}{\kappa_1 + 1} \frac{1}{\sin(\lambda\pi)} \frac{1 + \alpha}{1 - \beta^2} [1 - 2\beta(\lambda - 1)]$$

and

$$\tilde{b}(t) = b(t)(1 + t)^{\lambda}(1 - t)^{1/2}$$

is the regular part of the dislocation density ($\lambda = \frac{1}{2}$ when $\delta > 0$). For the numerical solution of (A1), two methods were compared. $\tilde{b}(t)$ in the first method is represented as a truncated series of Jacobi polynomials (Erdogan et al., 1973), while in the second method it is expressed in terms of piecewise quadratic polynomials (Miller and Keer, 1985). The results shown a faster convergence for the latter method for which 64 integration points were necessary to capture three significant figures, versus 400 integration points necessary for the first method.

Semi-infinite Crack Whose Tip is at Distance δ From the Interface

The singular integral equation is the same as the first (A2) except that the upper limit $2a$ is replaced with ∞. The unknown dislocation density is represented in real coordinates as

$$b(\xi) = \frac{k_I}{\sqrt{2\pi}} \frac{\xi^{1/2}}{(\xi - \delta)^{1/2}\xi^{\lambda}} \left[\frac{\kappa_1 + 1}{2\mu_1} \tilde{b}(\xi) + w(\xi)\psi^{-1} \right] \quad \text{(A5)}$$

with the additional condition $\lim_{\xi\to\infty} \tilde{b}(\xi) = 0$ replacing the crack closure condition that appears in the second Eq. (A2). As discussed by Rubinstein (1992), this representation stabilizes the singular integral equation. $w(\xi)$ is a function of the type

$$w(\xi) = \sin^2 \left[\frac{\pi}{2} \left(1 - \frac{\delta}{\xi} \right) \right]. \quad \text{(A6)}$$

With the change of variables $\xi = 2\delta/(1 - t)$, $y = 2\delta/(1 - \zeta)$ (A4) is normalized in the interval $[-1, 1]$.

By extracting the dominant term of the singularity of the resulting equation, the nondimensional ratio of the local and far-field stress intensity factors is determined as

$$f(\alpha, \beta) = \frac{K_I}{k_I} \delta^{\lambda - 1/2} = \tilde{b}(-1). \quad \text{(A7)}$$

Crack of Length $2a$ Extending Through the Interface by $\delta (\delta \geq 0)$

The set of coupled singular integral equations is given in (14) with A_i and K_{ij} defined in (A1). The compatibility condition at the interface is given in (15). The normalized nondimensional form of (14) and (15) is attained by means of the transformations

$$t_1 = \frac{\xi - a}{a}; \quad \zeta_1 = \frac{y - a}{a}; \quad \phi_1 = -\frac{\sigma^{(1)}(1 + \kappa_1)}{2\mu_1};$$

$$t_2 = -\frac{\xi + b}{b}; \quad \zeta_2 = -\frac{y + b}{b}; \quad \phi_2 = -\frac{\sigma^{(2)}(1 + \kappa_2)}{2\mu_2};$$

that lead to the following representation of the dislocation densities:

$$b^{(1)}(t_1) = \tilde{b}^{(1)}(t_1)(1 - t_1)^{-1/2}(1 + t_1)^{\mu};$$

$$b^{(2)}(t_2) = \tilde{b}^{(2)}(t_2)(1 - t_2)^{-1/2}(1 + t_2)^{\mu} \quad \text{(A8)}$$

where

$$c = b/a.$$

The nondimensional stress intensity factor at the crack tip closer to the interface is then given by

$$g^*(\alpha, \beta) = \frac{K_I}{\sigma^{(1)}\sqrt{\pi l}} = -\frac{1}{2^{\mu}\phi_1} \sqrt{\frac{2c}{1 + c}} \frac{1 + \alpha}{1 - \alpha} \tilde{b}^{(2)}(1). \quad \text{(A9)}$$

Semi-infinite Crack Whose Tip is at Distance δ Beyond the Interface

The equations for this case are similar to the first two Eqs. (14) with the upper limit $2a$ replaced with ∞, and Eq. (15). The third Eq. (14) is replaced by the following condition that stabilizes the integral equations and insures uniqueness of the solution:

$$\lim_{\xi\to\infty} \tilde{b}^{(1)}(\xi) = 0.$$

The dislocation density functions have the form

$$b^{(1)}(\xi) = \frac{k_I}{\sqrt{2\pi}} \frac{1}{(\xi + \delta)^{\lambda - \mu}\xi^{\mu}} \left[\frac{\kappa_1 + 1}{2\mu_1} \tilde{b}^{(1)}(\xi) + w_1(\xi)\psi^{-1} \right]$$

$$b^{(2)}(\xi) = \frac{k_I}{\sqrt{2\pi}} \frac{\kappa_2 + 1}{2\mu_2} \frac{\tilde{b}^{(2)}(\xi)}{(-\xi)^{\mu}(\delta + \xi)^{1/2}} \delta^{1/2 + \mu - \lambda} \quad \text{(A10)}$$

where $w_1(\xi)$ is a function of the type

$$w_1(\xi) = \sin^2 \left[\frac{\pi}{2} \frac{\xi + 1}{\xi} \right]. \quad \text{(A11)}$$

The normalized form of the set of equations is attained through the change of variables

$$t_1 = \frac{\xi - \delta}{\xi + \delta}; \quad \zeta_1 = \frac{y - \delta}{y + \delta}; \quad t_2 = \frac{2\xi + \delta}{\delta}; \quad \zeta_2 = \frac{2y + \delta}{\delta}.$$

By extracting the dominant term of the crack-tip singularity, the nondimensional ratio of the local and far-field stress intensity factor is then given by

$$f^*(\alpha, \beta) = \frac{K_I}{k_I} \delta^{\lambda - 1/2} = \tilde{b}^{(2)}(1). \quad \text{(A12)}$$

The crack-opening displacement at the interface is given by

$$\eta = -\int_{-\delta}^{0} B_2(\xi)d\xi = \frac{\delta}{2} \int_{-1}^{1} B_2(t_2)dt_2 = \frac{1 + \kappa_1}{2\mu_1} k_I \delta^{1-\lambda} \tilde{\eta}(\alpha, \beta) \quad \text{(A13)}$$

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