IMA Journal of Applied Mathematics (2011) **76**, 633−641
doi:10.1093/imamat/hxq070
Advance Access publication on January 7, 2011

Elastic stress diffusion around a thin corrugated inclusion

ROBERTO BALLARINI*
Department of Civil Engineering, University of Minnesota,
500 Pillsbury Drive SE. Minneapolis, MN 55455, USA
*Corresponding author: broberto@umn.edu

AND

PIERO VILLAGGIO
Dipartimento di Ingegneria Civile, Universita di Pisa, Via Carlo F. Gabba 22,
56122 Pisa, Italy

[Received on 18 February 2010]

We determine the stress state around a rigid slender body embedded within an infinite elastic medium when the body is pushed by a force acting along its longitudinal axis or when the body is unloaded but the medium is stretched at infinity. The problem is formulated in plane elasticity, where the complex variable method reduces it to an integral equation of Cauchy's type. If the profile of the inclusion is analytically representable by a polynomial, the solution is explicit.

Keywords: rigid inclusion; airfoil approximation; Cauchy integral.

1. Introduction

The problem of finding the state of stress around a rigid inclusion embedded within an infinite elastic medium is classic. The stresses can be generated either by applying tensions at infinity or by loading the inclusion by forces or couples. If the inclusion is a sphere or an ellipsoid, the solution is explicitly representable in terms of ellipsoidal spherical harmonics. The procedure is exposed in Lure (1964) where the author solves the case of an ellipsoidal cavity.

In plane elasticity, when the inclusion is a disk inserted in a hole perforating a plate, the class of explicit solutions is much larger. If the hole is circular, it is possible to consider various kinds of contact between the inclusion and the surrounding medium. For example, the contact may be frictionless or partial. Gladwell (1980) offers a wide account of the technically interesting cases and of the solution methods. But, in plane elasticity, exact solutions can be found even for inclusions of more general shape, provided that their boundaries may be conformally mapped into the unit circumference by a rational complex variable function. Milne-Thomson (1960) offers an elegant and simple procedure for treating the twisting of a rigid hypotrochoidal core, whose rim is welded to a hypotrochoidal hole.

Here, we study the case of a long but thin inclusion welded to an elastic plane and pushed by a force P acting along the longitudinal axis. The inclusion is symmetric with respect to this axis, and the lateral faces are two, symmetrically placed, curves of any shape but subjected to the only restriction of admitting a Cartesian analytic representation. We write the integral equation describing the problem and solve it under the assumption that the thickness of the inclusion is very small with respect to its longitudinal extent. Some particular solutions are illustrated and discussed. The same equation describes the case in which the inclusion is unloaded but the plate is stressed at infinity.

2. The integral equation of the elongated inclusion

Consider an infinite elastic plate of thickness h and denote its midplane by S (see Fig. 1). Choose a system of Cartesian x, y-axes such that the x-axis is horizontal and the y-axis is vertical. Take two points of abscissae ± 1 on the x-axis and consider a curve of equation $y(x)(-1 \leqslant x \leqslant 1)(y(x) \geqslant 0, y(\pm 1) = 0)$ connecting these points, and the curve $-y(x)$ mirror image of the former. The function $y(x)$ must satisfy the inequality $|y(x)| \ll 1$ and must be continuously differentiable along the interval $-1 \leqslant x \leqslant 1$. Suppose now that the region $-1 \leqslant x \leqslant 1$, $|y| \ll y(x)$, interior to the two curves is rigid and loaded by a force of magnitude P directed along the x-axis. The rigid core defined above introduces a stress state in the plate, and we want to determine it by exploiting the geometrical conditions $|y(x)| \ll 1$ and its consequent plausible approximations.

As a consequence of the symmetry of the inclusion with respect to the x-axis and the fact that P acts along the x-axis, the rigid inclusion will undergo a pure translation in the x-direction and the tractions at the upper and lower interfaces are symmetric. This simplifies the formulation of the problem for we can limit ourselves to determining the contact stress on the upper face of the inclusion.

The first consequence of the slenderness of the inclusion is that we may assume that the contact tractions at the interface $y = \pm y(x)$ to be practically horizontal and analytically defined by two symmetric distributions of tangential forces per unit length $p(y(x)) = p(-y(x))$ applied along the upper and lower face of the inclusion. A second consequence is that, due to the smallness of $|y(x)|$, the influence of these forces on the stress and strain state of the elastic infinite region around the inclusion is not sensibly altered if we replace them by a unique distribution of forces $2p(x)$ applied along the segment $-1 \leqslant x \leqslant 1$ (see Fig. 1). Extending a terminology used in aerodynamics, the present type of approximation may be called the 'airfoil' theory for inclusions (Szabó, 1964).

The problem is thus reduced to finding the function $p(x)$ by exploiting the geometric condition that the displacement state of the inclusion is a simple rigid translation in the x-direction. For this purpose, we first determine the stress components in an elastic infinite plate generated by a force $2p(x_0)\mathrm{d}x_0$ applied along a linear element $\mathrm{d}x_0$ containing the point x_0.

At a generic point of coordinates $z = x + iy$, the stress state has the form (Grigolyuk & Tolkachev, 1987)

$$\sigma_x + \sigma_y = -\frac{1+\nu}{2\pi h}\mathrm{Re}\int_{-1}^{1}\frac{2p(x_0)\mathrm{d}x_0}{z - x_0}, \tag{2.1}$$

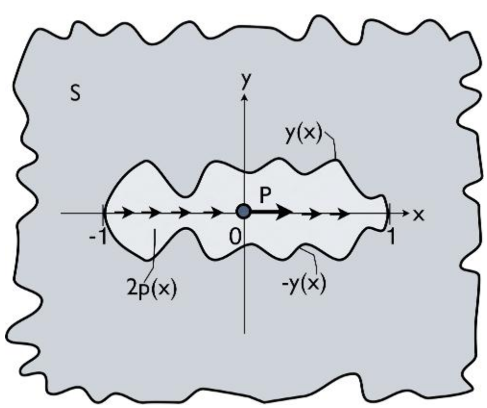

FIG. 1. Plate with a long but narrow inclusion.

$$\sigma_y - \sigma_x + 2i\tau_{xy} = \frac{1+\nu}{4\pi h}\int_{-1}^{1} 2p(x_0)\left[\frac{\bar{z}-x_0}{(z-x_0)^2} + \frac{3-\nu}{(1+\nu)(z-x_0)}\right]\mathrm{d}x_0, \tag{2.2}$$

where ν is Poisson's ratio.

Once we have the stresses, for the plane stress condition considered here, the corresponding strains are given by the constitutive equations

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y),$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x),$$

$$\varepsilon_{xy} = \frac{\tau_{xy}(1+\nu)}{E}, \tag{2.3}$$

where E is Young's modulus.

From (2.3), we can derive the strain component, ε_t, tangential to the curves $y = \pm y(x)$ whose unit tangent vectors are

$$n_x = \frac{1}{\sqrt{1+y'^2(x)}}, \quad n_y = \pm\frac{y'}{\sqrt{1+y'^2(x)}}. \tag{2.4}$$

The result is

$$\varepsilon_t(y(x)) = \varepsilon_x\frac{1}{1+y'^2} + \varepsilon_y\frac{y'^2}{1+y'^2} + 2\varepsilon_{xy}\frac{y'}{1+y'^2}, \tag{2.5}$$

$$\varepsilon_t(-y(x)) = \varepsilon_x\frac{1}{1+y'^2} + \varepsilon_y\frac{y'^2}{1+y'^2} - 2\varepsilon_{xy}\frac{y'}{1+y'^2}. \tag{2.6}$$

Not withstanding the apparent diversity, these two components are equal because $\varepsilon_{xy}(y(x)) = \varepsilon_{xy}(-y(x))$.

Since the adhesion along the interface is complete and the inclusion undergoes a rigid displacement in the x-direction, from (2.5) and (2.6), we can derive the equation

$$\varepsilon_t(y(x)) = \varepsilon_t(-y(x)) = 0, \tag{2.7}$$

which, after substitution of (2.3) and (2.1), (2.2), yields an integral equation for determining $p(x)$.

3. Approximate solution of (2.7)

Equation (2.7) is not explicitly solvable, but, through some approximations, it can be reduced to a tractable form. In terms of x, y (2.1) and (2.2) can be written as

$$\sigma_x + \sigma_y = -\frac{4(1+\nu)}{4\pi h}\int_{-1}^{1} p(x_0)\frac{x-x_0}{(x-x_0)^2+y^2}\mathrm{d}x_0, \tag{3.1}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = \frac{3-\nu}{4\pi h}\int_{-1}^{1} 2p(x_0)\frac{(x-x_0)-iy}{(x-x_0)^2+y^2}\mathrm{d}x_0$$
$$+ \frac{1+\nu}{4\pi h}\int_{-1}^{1} 2p(x_0)\frac{[(x-x_0)-iy]^3}{[(x-x_0)^2+y^2]^2}\mathrm{d}x_0. \tag{3.2}$$

Let us now exploit the assumptions $|y(x)| \ll 1$ and that $y(x)$ is continuously differentiable such that $|y'(\pm 1)| < \infty$. Then we make the approximations

$$
\begin{aligned}
(x - x_0)^2 + y^2 &\simeq (x - x_0)^2 \\
[(x - x_0) - iy]^3 &\simeq (x - x_0)^3 - 3iy(x - x_0)^2 \\
y = y(x) &\simeq y'(x_0)(x - x_0)
\end{aligned}
\tag{3.3}
$$

so that (3.1) and (3.2) may be replaced by

$$
\sigma_x + \sigma_y \simeq -\frac{2(1+\nu)}{2\pi h} \int_{-1}^{1} \frac{p(x_0)}{(x - x_0)} \mathrm{d}x_0, \tag{3.4}
$$

$$
\sigma_y - \sigma_x + 2i\tau_{xy} \simeq \frac{3-\nu}{2\pi h} \int_{-1}^{1} \frac{p(x_0)(1 - iy'(x_0))}{(x - x_0)} \mathrm{d}x_0 + \frac{1+\nu}{2\pi h} \int_{-1}^{1} p(x_0) \frac{1 - 3iy'(x_0)}{(x - x_0)} \mathrm{d}x_0. \tag{3.5}
$$

From (2.3), we derive the corresponding 'fundamental' strain combinations (the adjective has been introduced by Milne-Thomson, 1960):

$$
\varepsilon_x + \varepsilon_y = \frac{1}{E}(1-\nu)(\sigma_x + \sigma_y), \tag{3.6}
$$

$$
\varepsilon_y - \varepsilon_x + 2i\varepsilon_{xy} = \frac{1}{E}(1+\nu)(\sigma_y - \sigma_x + 2i\tau_{xy}) \tag{3.7}
$$

so that, after substitution of (3.4), (3.5) and isolation of the single strain components, we obtain

$$
\begin{aligned}
\varepsilon_x &= -\frac{(3-\nu)(1+\nu)}{2\pi E h} \int_{-1}^{1} \frac{p(x_0)}{x - x_0} \mathrm{d}x_0, \\
\varepsilon_y &= \frac{(1+\nu)^2}{2\pi E h} \int_{-1}^{1} \frac{p(x_0)}{x - x_0} \mathrm{d}x_0, \\
\varepsilon_{xy} &= -\frac{(3+\nu)(1+\nu)}{2\pi E h} \int_{-1}^{1} \frac{p(x_0) y'(x_0)}{x - x_0} \mathrm{d}x_0.
\end{aligned}
\tag{3.8}
$$

Therefore, (2.7) becomes

$$
\begin{aligned}
\frac{1+\nu}{2\pi E h} \frac{1}{1 + y'^2(x)} \Bigg[&-(3-\nu) \int_{-1}^{1} \frac{p(x_0)}{(x - x_0)} \mathrm{d}x_0 + (1+\nu) y'^2(x) \int_{-1}^{1} \frac{p(x_0)}{(x - x_0)} \mathrm{d}x_0 \\
&-2(3+\nu) y'(x) \int_{-1}^{1} \frac{p(x_0) y'(x_0)}{(x - x_0)} \mathrm{d}x_0 \Bigg] = 0,
\end{aligned}
\tag{3.9}
$$

which is equivalent to the integral equation

$$
\int_{-1}^{1} \frac{p(x_0)}{(x - x_0)} [-(3-\nu) + (1+\nu) y'^2(x) - 2(3+\nu) y'(x) y'(x_0)] \mathrm{d}x_0 = 0. \tag{3.10}
$$

But (3.10) is an integral equation of Cauchy's type that is explicitly integrable.

In particular, recalling the assumption $|y'(\pm 1)| < \infty$, the solution is unbounded at the points ± 1 and has the form (Grigolyuk & Tolkachev, 1987)

$$p(x) = \frac{C_0}{\sqrt{1-x^2}[(3-\nu)+(5+\nu)y'^2(x)]}, \tag{3.11}$$

where C_0 is an arbitrary constant determined by the condition of equilibrium

$$P = \int_{-1}^{1} 2p(x)\mathrm{d}x. \tag{3.12}$$

In order to illustrate the result, consider the case in which $y(x)$ is an arc of parabola of equation $y(x) = \varepsilon(1-x^2)$, where ε is a small constant. Then (3.11) becomes

$$p(x) = \frac{C_0}{\sqrt{1-x^2}[(3-\nu)+4\varepsilon^2(5+\nu)x^2]} \tag{3.13}$$

and computation of integral (3.12) yields (Gradshteyn & Ryzhik, 1965)

$$C_0 = \frac{P}{2\pi}\sqrt{(3-\nu)(3-\nu+4\varepsilon^2(5+\nu))} \tag{3.14}$$

so that the complete expression of $p(x)$ is

$$p(x) = \frac{P}{2\pi}\frac{\sqrt{(3-\nu)(3-\nu+4\varepsilon^2(5+\nu))}}{\sqrt{1-x^2}[3-\nu+4\varepsilon^2(5+\nu)x^2]}. \tag{3.15}$$

Note that for $\varepsilon = 0$, when the inclusion reduces to a plane rigid lamina, (3.15) recovers the classical distribution $p(x) = \frac{P}{2\pi\sqrt{1-x^2}}$ recorded in several treatises on plane elasticity (Grigolyuk & Tolkachev, 1987).

4. The optimal inclusion

Formula (3.11) furnishes us the distribution of $p(x)$ once $y'(x)$ is given. But the problem may be inverted. Is there a shape of the $\pm y(x)$ boundaries such that $p(x)$ has a prescribed distribution? In particular, is there a pair $\pm y(x)$ that maximize the total force P calculated according to (3.12)? Mathematically formulated, the problem consists in finding a function $\pm y(x)$ maximizer of the functional

$$P = \int_{-1}^{1} p(x)\mathrm{d}x = 2C_0\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}[3-\nu+(5+\nu)y'^2(x)]} \tag{4.1}$$

with the boundary conditions $y(\pm 1) = 0$. The Euler's equation of the variational problem is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2(5+\nu)y'(x)}{\sqrt{1-x^2}[3-\nu+(5+\nu)y'^2(x)]^2}\right] = 0 \tag{4.2}$$

and a first integral of (4.2) is

$$2(5+\nu)y'(x) = A\sqrt{1-x^2}[(3-\nu)+(5+\nu)y'^2(x)]^2, \tag{4.3}$$

where A is a constant. Without proceeding further, it is immediate to recognize that, putting $A = 0$, the solution to (4.3), satisfying the boundary conditions $y(\pm 1) = 0$, is $y(x) \equiv 0$. Thus, we get the intuitively expected result that the best inclusion is a rectilinear lamina.

But this result may not be satisfactory because, in general, the inclusion is a deep solid with a prescribed volume. Therefore, a more realistic formulation of the problem is that of adding to the problem of maximizing (3.12) the isoperimetrical restriction

$$\int_{-1}^{1} y(x)\mathrm{d}x = \text{const} = (\text{say}) = 1. \tag{4.4}$$

We apply the method of Lagrange's multipliers, and, after standard calculations, we obtain the following Euler's equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2(5+\nu)y'(x)}{\sqrt{1-x^2}[3-\nu+(5+\nu)y'^2(x)]^2}\right] + \lambda = 0 \tag{4.5}$$

with the boundary conditions $y(\pm 1) = 0$. In this case, an explicit solution is not available. But a reasonable approximation is that of disregarding the term $y'^2(x)$ in the denominator of (4.5). After this simplification, (4.5) becomes

$$\frac{2(5+\nu)}{(3-\nu)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'(x)}{\sqrt{1-x^2}}\right] + \lambda = 0 \tag{4.6}$$

whose general integral is (Gradshteyn & Ryzhik, 1965)

$$\frac{2(5+\nu)}{(3-\nu)}y(x) = A\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right) + \frac{\lambda}{3}\left(\sqrt{1-x^2}\right)^3 + B, \tag{4.7}$$

where A and B are two arbitrary constants. The boundary conditions $y(\pm 1) = 0$ imply $A = B = 0$, and hence the solution assumes the surprisingly simple form

$$y(x) = y_{\text{opt}}(x) = \lambda'\left(\sqrt{1-x^2}\right)^3 \tag{4.8}$$

having put $\lambda' = \frac{(3-\nu)}{6(5+\nu)}\lambda$. The constant λ' is determined by the isoperimetric condition (4.4), and a simple integration yields (Gradshteyn & Ryzhik, 1965)

$$1 = \lambda'\int_{-1}^{1}\left(\sqrt{1-x^2}\right)^3 \mathrm{d}x = \frac{3}{8}\pi\lambda', \tag{4.9}$$

hence $\lambda' = \frac{8}{3\pi}$.

5. A numerical comparison

An instructive indication about the advantage of adopting the profile (4.8) is offered by comparing the magnitude of the force P associated to it with the value of P associated to a parabolic profile like that described by $y(x) = \varepsilon(1-x^2)$ discussed in Section 3. In order to do this, we first calculate the area subtended by this curve, namely $\int_{-1}^{1}\varepsilon(1-x^2)\mathrm{d}x = \frac{4}{3}\varepsilon$. The curve of equation (4.8) subtending the

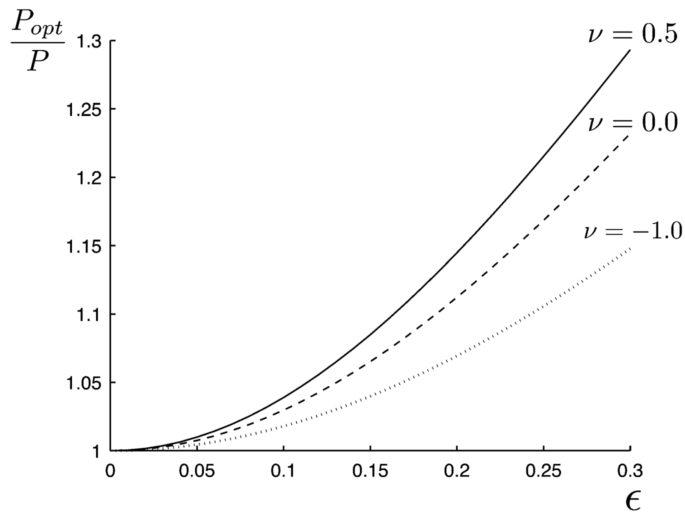

FIG. 2. Optimal force on rigid inclusion.

same area is therefore determined by replacing 1 by $\frac{4}{3}\varepsilon$ in the left-hand side of (4.9). The result is that we must put $\lambda' = \frac{32}{9}\pi\varepsilon$ instead of $\lambda' = \frac{8}{3}\pi$.

If we assign C_0, the value given by (3.14), the parabolic profile yields, of course, the result that the resultant is P. On the other hand, we choose the profile having (4.8), with $\lambda' = \frac{32}{9}\pi\varepsilon$ and consider the distribution of forces per unit length

$$p_{\rm opt}(x) = \frac{C_0}{\sqrt{1-x^2}[(3-\nu)+(5+\nu)y'^2_{\rm opt}(x)]}, \tag{5.1}$$

where C_0 is given by (3.14). Differentiation of (4.8), substitution into (5.1), and successive integration yields the value

$$\begin{aligned} P_{\rm opt} &= 2\int_{-1}^{1} p_{\rm opt}(x){\rm d}x = 2\int_{-1}^{1}\frac{P}{2\pi}\frac{\sqrt{(3-\nu)[3-\nu+4\varepsilon^2(5+\nu)]}{\rm d}x}{\sqrt{1-x^2}[3-\nu+(\frac{32}{9}\pi)\varepsilon^2 9x^2(1-x^2)]} \\ &= P\sqrt{\frac{3-\nu+4\varepsilon^2(5+\nu)}{3-\nu+\frac{32}{9\pi^2}\varepsilon^2(5+\nu)}}. \end{aligned} \tag{5.2}$$

Since $4 > \frac{32}{9\pi^2}$, we see from (5.2) that $P_{\rm opt} > P$ for any value of ν $\left(-1 < \nu < \frac{1}{2}\right)$, and as shown in Fig. 2, the discrepancy is remarkable even for small values of ε.

6. The integral equation of an inclusion welded to a stretched plate

The same technique, suitably adjusted, can be used for solving the elastic problem of an elongated rigid inclusion inserted into an elastic plate subject to a prescribed state of uniform stress at infinity. In particular, assume that the rigid slender stiffener of Section 2 is not loaded by a longitudinal force P, but is welded to a plate stressed at infinity by constant stresses σ_x^0, σ_y^0 and τ_{xy}^0.

In abscence of the stiffener, the corresponging strains would be

$$\varepsilon_x^0 = \frac{1}{E}(\sigma_x^0 - \nu\sigma_y^0),$$

$$\varepsilon_y^0 = \frac{1}{E}(\sigma_y^0 - \nu\sigma_x^0),$$

$$\varepsilon_{xy}^0 = \frac{\tau_{xy}^0(1+\nu)}{E} \tag{6.1}$$

and the strain tangential to the curves $y = \pm y(x)$ would assume the form (cf. (2.5))

$$\varepsilon_t^0 = \varepsilon_x^0 \frac{1}{1+y'^2} + \varepsilon_y^0 \frac{y'^2}{1+y'^2} + 2\varepsilon_{xy}^0 \frac{y'}{1+y'^2}. \tag{6.2}$$

This strain is prevented by the inclusion that exerts a shear force $p(x_0)$ on the plate and a tangential strain ε_t, where ε_t is just the left-hand side of (3.9). Since the total tangential strain at the interface is zero, we have the equation $\varepsilon_t + \varepsilon_t^0 = 0$, which after use of (3.8) and some simplifications, yields the following integral equation for $p(x_0)$:

$$\frac{1+\nu}{2\pi Eh}\int_{-1}^{1} \frac{p(x_0)}{(x-x_0)}[-(3-\nu) + (1+\nu)y'^2(x) - 2(3+\nu)y'(x)y'(x_0)]\mathrm{d}x_0$$
$$+ \frac{1}{E}[(\sigma_x^0 - \nu\sigma_y^0) + (\sigma_y^0 - \nu\sigma_x^0)y'^2(x) + 2\tau_{xy}^0(1+\nu)y'(x)] = 0. \tag{6.3}$$

This is again an equation of Cauchy's type, but non-homogeneous: therefore, the solution is the sum of the general integral (3.11) plus a particular integral obtained by a standard procedure (Grigolyuk & Tolkachev, 1987). The result is

$$p(x) = \frac{1}{[(3-\nu) + (5+\nu)y'^2(x)]}\left\{\frac{C_0}{\sqrt{1-x^2}} - \frac{2\pi h}{(1+\nu)}\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-x_0^2}}{(x-x_0)}[(\sigma_x^0 - \nu\sigma_y^0)\right.$$
$$\left. + (\sigma_y^0 - \nu\sigma_x^0)y'^2(x_0) + 2\tau_{xy}^0(1+\nu)y'(x_0)]\mathrm{d}x_0\right\}. \tag{6.4}$$

The arbitrary constant C_0 is now determined by the condition that the resultant of forces $p(x)$ applied on the two sides of the inclusion vanishes

$$\int_{-1}^{1} 2p(x)\mathrm{d}x = 0. \tag{6.5}$$

As an example, if $y(x)$ is an arc of parabola of equation $y(x) = \varepsilon(1-x^2)$, where $\varepsilon \ll 1$, (6.5), after neglection of the terms containing ε^2 becomes

$$C_0\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} - \frac{2h}{\pi(1+\nu)}\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}}\int_{-1}^{1}[(\sigma_x^0 - \nu\sigma_y^0) \tag{6.6}$$
$$-4\tau_{xy}^0(1+\nu)\varepsilon x_0]\frac{\sqrt{1-x_0^2}}{(x-x_0)}\mathrm{d}x_0 = 0.$$

An application of the residuum theory yields the result

$$\int_{-1}^{1}[(\sigma_x^0-\nu\sigma_y^0)-4\tau_{xy}^0(1+\nu)\varepsilon x_0]\frac{\sqrt{1-x_0^2}}{(x-x_0)}\mathrm{d}x_0 = \pi\left[(\sigma_x^0-\nu\sigma_y^0)x-4\tau_{xy}^0(1+\nu)\varepsilon\left(x^2-\frac{1}{2}\right)\right]. \tag{6.7}$$

But, since a second integration shows that

$$\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}}\left[(\sigma_x^0-\nu\sigma_y^0)x-4\tau_{xy}^0(1+\nu)\varepsilon\left(x^2-\frac{1}{2}\right)\right]=0, \tag{6.8}$$

we conclude that $C_0=0$, and hence the distribution of the forces $p(x)$ is

$$p(x)=\frac{2h}{(3-\nu)(1+\nu)\sqrt{1-x^2}}\left[(\sigma_x^0-\nu\sigma_y^0)x-4\tau_{xy}^0(1+\nu)\varepsilon\left(x^2-\frac{1}{2}\right)\right]. \tag{6.9}$$

For $\varepsilon=0$, we obtain the distribution of forces $p(x)$ at the interfaces of a plane lamina.

Funding

R. B. is grateful for support from the James L. Record Chair.

REFERENCES

GLADWELL, G. (1980) *Contact Problems in the Classical Theory of Elasticity.* Alphen aan der Rijn, The Netherlands: Sijthoff and Noordhoff International Publishers.

GRADSHTEYN, I. & RYZHIK , I. (1965) *Table of Integrals Series and Products.* New York: Academic Press.

GRIGOLYUK, E. & TOLKACHEV, V. (1987) *Contact Problems in the Theory of Plates and Shells.* Moscow: Mir Publishers.

LURE, A. I. (1964) *Three-dimensional Problems in the Theory of Elasticity.* Groningen, The Netherlands: Interscience Publishers.

MILNE-THOMSON, L. (1960) *Plane Elastic Systems.* Berlin: Springer.

SZABÓ, I. (1964) *Hhere Technische Mechanik.* Berlin: Springer.