International Journal of Fracture 33: R23–R26 (1987)

AN INTEGRAL EQUATION APPROACH FOR RIGID LINE INHOMOGENEITY PROBLEMS

R. Ballarini
Department of Civil Engineering, Case Western Reserve University
Cleveland, Ohio 44106 USA
tel: (216) 368-2963

The characteristics of the stress field near the tips of a rigid line inhomogeneity have been reported by Chou [1] and Wang et al. [2]. They obtained the solution to the problem of a rigid line inhomogeneity under the action of an inclined unidirectional loading by taking the solution of an elliptical inhomogeneity and letting the ratio of the minor to major semiaxes approach zero. In their investigation the problem was solved using two methods: Eshelby's equivalent inclusion method, and conformal mapping of Muskhelishvili's complex potentials. The results showed that the stresses at the tips of the inhomogeneity are square root singular (as in the case of the corresponding crack problem), and that the stress intensity factors depend on Poisson's ratio.

In the present paper a more direct method for arriving at the solution to the problem of a rigid line inhomogeneity is presented. The problem is formulated in terms of singular integral equations, and can therefore be extended to cases where the line defining the inhomogeneity is curved. Moreover, this approach clearly illustrates the similarities between crack problems and rigid line inhomogeneity problems. This method has been applied by the author to the study of the failure mechanisms of rigid anchors embedded in brittle materials [3,4].

The elasticity problem shown in Fig. 1 is formulated in terms of the complex potentials of Muskhelishvili [5]. The stresses and displacements can be expressed in terms of the analytic functions Φ and Ψ as

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathit{Real}\,[\Phi(z)] \tag{1}$$

$$\sigma_{yy} - i\,\sigma_{xy} = \Phi(z) + \overline{\Phi(z)} + z\overline{\Phi'(z)} + \overline{\Psi(z)} \tag{2}$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \kappa\,\Phi(z) - \overline{\Phi(z)} - z\overline{\Phi'(z)} - \overline{\Psi(z)} \tag{3}$$

where $i=\sqrt{-1}$, $z=x+iy$, μ is the shear modulus, $\kappa=3-4\nu$ for plane strain, and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν being Poisson's ratio. Primes denote differential with respect to z, and bars imply complex conjugation. The boundary conditions along the rigid line inhomogeneity are

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = 0 \tag{4}$$

The solution is sought in the form

$$\Phi(z) = \sigma_\infty/4 + \int_{-a}^{+a} \frac{\alpha(\xi)\, d\xi}{z-\xi} \tag{5}$$

$$\Psi(z) = \sigma_\infty/2 - \kappa\int_{-a}^{+a} \frac{\alpha(\xi)\, d\xi}{z-\xi} + \int_{-a}^{+a} \frac{\xi\alpha(\xi)\, d\xi}{(z-\xi)^2} \tag{6}$$

where σ_∞ represents the far field stress and $\alpha(\xi)$ is a distribution of body forces defined by

$$\alpha(\xi) = \frac{-1}{2\pi(\kappa+1)} \frac{\partial}{\partial\xi} (F_x+iF_y) \tag{7}$$

with F_x and F_y representing the x and y components of the force, respectively. Substitution of (5) and (6) into (3), and enforcing (4) leads to

$$2\kappa \int_{-a}^{+a} \frac{\alpha(\xi)\, d\xi}{\xi - x} = \sigma_\infty(\kappa-3)/4 \qquad -a \leq x \leq +a \tag{8}$$

To ensure uniqueness of the solution to (8) the following condition (zero net force on the inhomogeneity) must be satisfied

$$\int_{-a}^{+a} \alpha(\xi)\, d\xi = 0 \tag{9}$$

We note that for a crack (subjected to a uniform far field tensile stress) modeled as a continuous distribution of dislocations the governing integral equations for the dislocation density are

$$2\int_{-a}^{+a} \frac{\beta(\xi)\, d\xi}{(\xi-x)} = \sigma_\infty \qquad -a \leq x \leq +a \tag{10}$$

$$\int_{-a}^{+a} \beta(\xi)\, d\xi = 0 \tag{11}$$

where the dislocation density is defined by

$$\beta(\xi) = \frac{\mu e^{i\theta}}{\pi i(\kappa+1)} \frac{\partial}{\partial\xi} \{ [u_r] + i[v_\theta] \} \tag{12}$$

with $[u_r]$ and $[v_\theta]$ representing the magnitudes of the displacement jumps. The similarity between the inhomogeneity problem and the crack problem can be clearly seen by comparing (8) and (10). The solution of (8) and (9) can be obtained in closed form using the techniques in [5]. It is given by

$$\beta(\xi) = \frac{\mu e^{i\theta}}{\pi i(\kappa+1)} \frac{\partial}{\partial\xi} \{[u_r] + i[v_\theta]\} \tag{12}$$

with $[u_r]$ and $[v_\theta]$ representing the magnitudes of the displacement jumps. The similarity between the inhomogeneity problem and the crack problem can be clearly seen by comparing (8) and (10). The solution of (8) and (9) can be obtained in closed form using the techniques in [5]. It is given by

$$\alpha(x) = \sigma_\infty \frac{(\kappa-3)}{8\kappa\pi} \frac{x}{(a^2- x^2)^{\frac{1}{2}}} \tag{13}$$

The complex potentials can be obtained by substituting (13) into (5) and (6), and performing the integration. The stresses and displacements can then be obtained from the potentials through (1)-(3). In particular, the stress intensity factors, defined by

$$K_I - i\, K_{II} = \lim_{x\to a} \sqrt{2\pi(x-a)} \quad (\sigma_{yy}-i\sigma_{xy}) \tag{14}$$

become

$$K_I = \sigma\sqrt{(\pi a)}\, \frac{(\kappa-1)\ (3-\kappa)}{8\kappa} \qquad K_{II} = 0 \tag{15}$$

The result agrees with that obtained by Wang and shows that the stress intensity factor depends on Poisson's ratio. (Note that for $\nu=0$ or $\nu=.5$ the stress intensity factor is zero).

For the corresponding crack problem $K_I = \sigma\sqrt{(\pi a)}$, therefore the ratio of stress intensity factors for the two problems is

$$\frac{K_I\ (\text{inhomogeneity})}{K_I(\text{crack})} = (\kappa-1)\ (3-\kappa)/8\kappa \tag{16}$$

We note that this method could be easily extended to the case of curved rigid line inhomogeneities and/or finite geometries by modifying (4)-(6) using the same techniques that are employed to solve crack problems.

REFERENCES

[1] Y.T. Chou and Z.Y. Wang, in *Recent Development in Applied Mathematics*, Rensselaer Press (1983) 21-30.

[2] Z.Y. Wang, H.T. Zhang, and Y.T. Chou, *Journal of Applied Mechanics* 52 (1985) 818-822.

[3] R. Ballarini, L.M. Keer, and S.P. Shah, *International Journal of Fracture* 33 (1987) to appear.

[4] R. Ballarini, S.P. Shah, and L.M. Keer, *Proceedings of the Royal Society* (London) A404 (1986) 35-54.

[5] N.I. Muskhelishvili, *Some Basic Problems of Mathematical Theory of*

Elasticity, Noordhoff Ltd., Groningen (1953).

5 January 1987

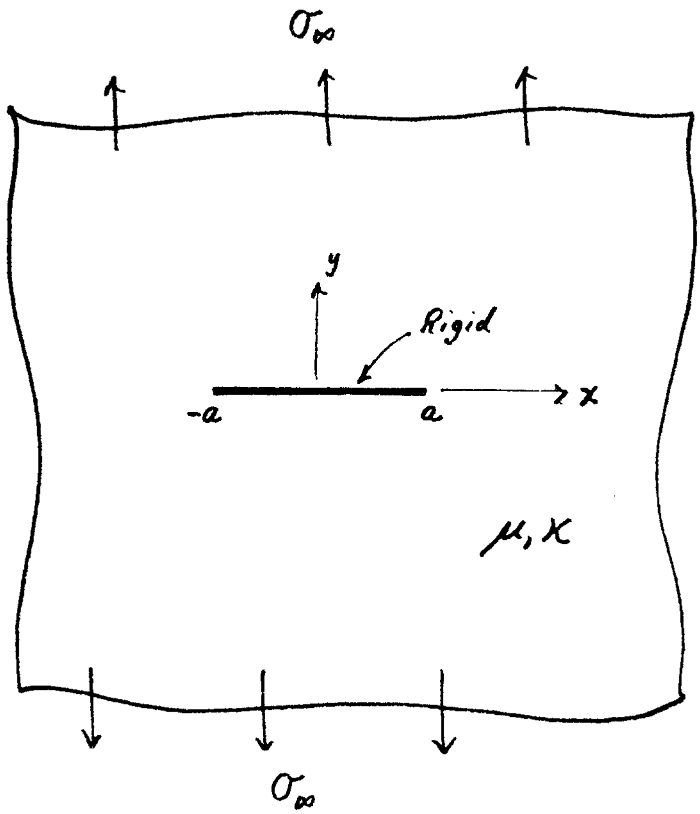

Figure 1. Rigid line inhomogeneity under tensile loading.