International Journal of Fracture (2006) 139:59–69
DOI 10.1007/s10704-006-6730-0

Frobenius' method for curved cracks

ROBERTO BALLARINI[1,*] and PIERO VILLAGGIO[2]
[1] *Department of Civil Engineering, Case Western Reserve University, Cleveland, OH 44106-7201, USA*
[2] *Dipartimento di Ingegneria Strutturale, Universitá di Pisa, Pisa, Italy*
[*] *Author for Correspondence (E-mail: roberto.ballarini@case.edu)*

Received 10 November 2005; accepted in revised form 11 January 2006

Abstract. The distribution of stresses produced by an undulated crack in a plane elastic solid, and in particular, at its tips where stresses approach infinity, requires the solution of two coupled singular integral equations. Except for simple crack geometries such as rectilinear and circular arcs in infinite plates, for which explicit analytic solutions have been obtained, the integral equations require numerical solutions. We propose a treatment of the integral equations by Frobenius' method, which is particularly suitable for evaluating the stress intensity factors of slightly curved cracks.

Key words: Curved crack, Frobenius' method, stress intensity factors.

1. Introduction

The stresses induced by a curved crack in an infinite elastic sheet subjected to a uniform stress state at infinity can be obtained from the dislocation density, which satisfies two coupled singular integral equations. Explicit analytical solutions of these equations have been obtained only for rectilinear and circular cracks. But often the shape of the crack is wavy and hence numerical solution of the integral equations is required to determine the dislocation density, the stress field and the stress intensity factors. For finite length curved cracks Chen et al. (1991) proposed a method of solution of the integral equations that involves representing the regular part of the dislocation density as a linear combination of Chebyshev polynomials, whose coefficients are determined by collocation. The same procedure was proposed by Dreilich and Gross (1985) for the specific case of a slightly curved crack. Brandinelli (1997) solved the same problems by approximating the dislocation density with Chebyshev polynomials. The numerical values of the stress intensity factors calculated for parabolic and sinusoidal shaped cracks were, to within three significant figures, the same as those obtained by Savruk (1981) using a different method.

An elegant treatment of the problem of slightly curved and kinked cracks involving simultaneous analytical perturbation of the (given) boundary of the crack and of the unknown complex stress functions (Muskhelishvili, 1953) was presented by Cotterell and Rice (1980), who showed that satisfactory values of the stress intensity factors can be obtained even by arresting the perturbation procedure to first order.

The integral equations of the curved crack can also be solved by the method of power series (Frobenius' method) consisting of a double expansion, in increasing powers of a suitable curvilinear abscissa, of the equation of the crack and of the dislocation density, followed by a term-by-term equalization of quantities of the same powers in the abscissa. Frobenius' method becomes particularly effective for relatively 'flat' cracks, in which the coefficients of the polynomial expansion are so small that their products can be neglected without appreciable error. In this case the method yields a system of infinite linear algebraic equations containing the coefficients of the regular part of the dislocation density as unknowns. Once these coefficients have been determined, the evaluation of all physical quantities including the stress intensity factors is immediate.

In this paper we apply Frobenius' method to flat cracks. This may appear a severe restriction on the practical usefulness of the method, since cracks with large undulations and sharp corners violate the assumption of small coefficients in the polynomial representation of the crack. But Frobenius' method is sufficiently flexible for being extended outside its theoretical range of applicability. In particular, we have calculated the stress intensity factor at the tips of a parabolic and of a so called 'snake crack' (a third order polynomial) comparing the results with those obtained by Chen et al. (1991), Brandinelli (1997), and Cotterell and Rice (1980).

2. The integral equation of the curved crack

We consider an infinite plate subject to a uniform stress state at infinity containing a curved crack of finite extent (Figure 1). With reference to a x, y-Cartesian system of coordinates we denote the remote stresses by $\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty$, respectively. We assume the profile of the crack to be defined by the polynomial equation

$$y(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots \quad (0 \leqslant x \leqslant 1)\,. \tag{2.1}$$

The displacement discontinuities across the surfaces of the crack, that alter the fundamental state of stress, can be analytically described using the Green's functions of dislocations with components b_x, b_y, which in turn are distributed along the curve (2.1). According to Dundurs and Sendeckyj (1965) a discrete dislocation at a point (ξ, η) of the curve generates at (x, y) a stress state defined by the Airy stress function

$$U = \frac{2\mu}{\pi\,(\kappa+1)}\left[b_y\,(x-\xi)\ln r - b_x\,(y-\eta)\ln r\right], \tag{2.2}$$

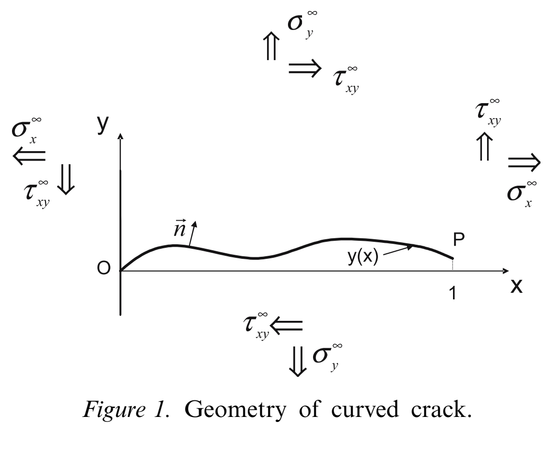

Figure 1. Geometry of curved crack.

where r $\left(r=\sqrt{(x-\xi)^2+(y-\eta)^2}\right)$ denotes the distance between (x, y) and (ξ, η), μ is the shear modulus, $\kappa=3-4\nu$ for plane strain, $\kappa=\frac{3-\nu}{1+\nu}$ for plane stress, and ν is Poisson's ratio. For convenience we define $B_i=\frac{2\mu}{\pi(\kappa+1)}b_i$ $(i=x, y)$, $x_1=x-\xi$, $y_1=y-\eta$, and by differentiation obtain the stress components

$$\sigma_x^1=\frac{\partial^2 U}{\partial y^2}=B_y\left(\frac{x_1}{r^2}-2\frac{x_1 y_1^2}{r^4}\right)-B_x\left(3\frac{y_1}{r^2}-2\frac{y_1^3}{r^4}\right), \tag{2.3a}$$

$$\sigma_y^1=\frac{\partial^2 U}{\partial x^2}=B_y\left(3\frac{x_1}{r^2}-2\frac{x_1^3}{r^4}\right)-B_x\left(\frac{y_1}{r^2}-2\frac{x_1^2 y_1}{r^4}\right), \tag{2.3b}$$

$$\tau_{xy}^1=\frac{-\partial^2 U}{\partial x \partial y}=-B_y\left(\frac{y_1}{r^2}-2\frac{x_1^2 y_1}{r^4}\right)+B_x\left(\frac{x_1}{r^2}-2\frac{x_1 y_1^2}{r^4}\right). \tag{2.3c}$$

The dislocation components B_x, B_y are replaced by distributions along the arc OP (Figure 1), defined in terms of the displacement discontinuities $[u_i]$ $(i=x, y)$ as $B_i \equiv \frac{\partial[u_i]}{\partial s}$, and hence the total stress state at (x, y) is given by the integrals

$$\sigma_x=\int_0^P \sigma_x^1 \,\mathrm{d}s, \quad \sigma_y=\int_0^P \sigma_y^1 \,\mathrm{d}s, \quad \tau_{xy}=\int_0^P \tau_{xy}^1 \,\mathrm{d}s, \tag{2.4}$$

where $\mathrm{d}s$ is the arc element. With ξ as a parameter, the element $\mathrm{d}s$ can be written as $\mathrm{d}s=\sqrt{1+y'^2}\mathrm{d}\xi$, and (2.4) become

$$\sigma_x=\int_0^1 \sigma_x^1\sqrt{1+y'^2}\,\mathrm{d}\xi, \quad \sigma_y=\int_0^1 \sigma_y^1\sqrt{1+y'^2}\,\mathrm{d}\xi, \quad \tau_{xy}=\int_0^1 \tau_{xy}^1\sqrt{1+y'^2}\,\mathrm{d}\xi. \tag{2.5}$$

The total stress at any point (x, y) along the curve OP is the sum of the fundamental state $\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty$ and the dislocation state (2.4) (or (2.5)). The zero-traction condition along the crack surfaces is enforced through the equations

$$\left(\sigma_x^\infty+\sigma_x\right)n_x+\left(\tau_{xy}^\infty+\tau_{xy}\right)n_y=0, \tag{2.6a}$$

$$\left(\tau_{xy}^\infty+\tau_{xy}\right)n_x+\left(\sigma_y^\infty+\sigma_y\right)n_y=0, \tag{2.6b}$$

where n_x, n_y are the components of the unit vector $\vec{n}$ normal to the curve OP. Let us note that, since OP admits the Cartesian representation, $y=y(x)$, the components of $\vec{n}$ have the expressions

$$n_x=-\frac{y'}{\sqrt{1+y'^2}}, \quad n_y=\frac{1}{\sqrt{1+y'^2}}. \tag{2.7}$$

Equations (2.6), combined with (2.3), (2.4), form a system of two coupled singular integral equations in the two unknown functions B_x, B_y. The condition that displacements be single-valued (the crack is closed at both ends) introduces two other restrictions on the unknowns

$$\int_0^P B_i \,\mathrm{d}s=\int_0^1 B_i\sqrt{1+y'^2}\,\mathrm{d}\xi=0. \tag{2.8}$$

3. The simplified treatment of the flat crack

So far no approximation has been made in writing the integral equations for B_x, B_y. Let us now assume that coefficients A_i in (2.1) are so small compared to unity that their products are negligible. Retaining only the first three terms of the expansion (2.1), we obtain greatly simplified expressions for the stress components (2.3)

$$\sigma_x^1 \cong \frac{B_y}{x-\xi} - 3\frac{B_x}{x-\xi}\left[A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right], \tag{3.1a}$$

$$\sigma_y^1 \cong \frac{B_y}{x-\xi} + \frac{B_x}{x-\xi}\left[A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right], \tag{3.1b}$$

$$\tau_{xy}^1 \cong \frac{B_x}{x-\xi} + \frac{B_y}{x-\xi}\left[A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right]. \tag{3.1c}$$

Consistent with this approximation, the components (n_x, n_y) become

$$n_y \cong 1, \quad n_x \cong -(A_1 + 2A_2x + 3A_3x^2), \tag{3.2}$$

and the contribution of the fundamental state in (2.6) is

$$T_x = \sigma_x^\infty n_x + \tau_{xy}^\infty n_y = -(A_1 + 2A_2x + 3A_3x^2)\sigma_x^\infty + \tau_{xy}^\infty, \tag{3.3a}$$

$$T_y = \tau_{xy}^\infty n_x + \sigma_y^\infty n_y = -(A_1 + 2A_2x + 3A_3x^2)\tau_{xy}^\infty + \sigma_y^\infty. \tag{3.3b}$$

Then, combining Equation (3.1) with (3.3), we obtain the following system of integral equations of Cauchy type

$$-(A_1 + 2A_2x + 3A_3x^2)\int_0^1 \left[\frac{B_y}{x-\xi} - 3\frac{B_x}{x-\xi}\left\{A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right\}\right] d\xi$$
$$+\int_0^1 \left[\frac{B_x}{x-\xi} + \frac{B_y}{x-\xi}\left\{A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right\}\right] d\xi + T_x = 0, \tag{3.4a}$$

$$-(A_1 + 2A_2x + 3A_3x^2)\int_0^1 \left[\frac{B_x}{x-\xi} - \frac{B_y}{x-\xi}\left\{A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right\}\right] d\xi$$
$$+\int_0^1 \left[\frac{B_y}{x-\xi} + \frac{B_x}{x-\xi}\left\{A_1 + A_2(x+\xi) + A_3(x^2+x\xi+\xi^2)\right\}\right] d\xi + T_y = 0. \tag{3.4b}$$

But here again we can neglect terms containing products of the type $A_i A_j$ $(i, j = 1, 2, 3)$ and reduce (3.4) to the form

$$\int_0^1 \left[\frac{B_y}{x-\xi}\left\{-A_2x - 2A_3x^2 + A_2\xi + A_3x\xi + A_3\xi^2\right\} + \frac{B_x}{x-\xi}\right] d\xi + T_x = 0, \tag{3.5a}$$

$$\int_0^1 \left[\frac{B_x}{x-\xi}\left\{-A_2x - 2A_3x^2 + A_2\xi + A_3x\xi + A_3\xi^2\right\} + \frac{B_y}{x-\xi}\right] d\xi + T_y = 0. \tag{3.5b}$$

At the same time the unknowns B_i must satisfy the compatibility equations (2.8), which, after linearization, become

$$\int_0^1 B_i \,\mathrm{d}\xi = 0. \tag{3.6}$$

4. Application of Frobenius' method

In order to solve (3.5), (3.6) we start from a plausible representation of B_x, B_y that is consistent with the square root singularity in dislocation densities at both crack tips

$$B_x(\xi) = \frac{1}{\sqrt{\xi(1-\xi)}}(B_1 + B_2\xi + B_3\xi^2 + B_4\xi^3 + \cdots), \tag{4.1a}$$

$$B_y(\xi) = \frac{1}{\sqrt{\xi(1-\xi)}}(C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 + \cdots), \tag{4.1b}$$

where $B_1, \ldots, C_1, \ldots$ are constants to be determined. Substitution of (4.1) in (3.5), (3.6) and the application of the residuum theory (Grigolyuk and Tolkachev (1987)) leads to the following identities

$$\int_0^1 \frac{B_x(\xi)\mathrm{d}\xi}{x-\xi} = \int_0^1 \frac{B_1 + B_2\xi + B_3\xi^2 + B_4\xi^3}{\sqrt{\xi(1-\xi)}(x-\xi)}\mathrm{d}\xi$$
$$= -\pi\left[B_2 + \left(x + \frac{1}{2}\right)B_3 + \left(x^2 + \frac{x}{2} + \frac{3}{8}\right)B_4\right],$$

$$\int_0^1 \frac{B_x(\xi)\xi\mathrm{d}\xi}{x-\xi} = \int_0^1 \frac{B_1\xi + B_2\xi^2 + B_3\xi^3 + B_4\xi^4}{\sqrt{\xi(1-\xi)}(x-\xi)}\mathrm{d}\xi$$
$$= -\pi\left[B_1 + \left(x + \frac{1}{2}\right)B_2 + \left(x^2 + \frac{x}{2} + \frac{3}{8}\right)B_3 + \left(x^3 + \frac{x^2}{2} + \frac{3x}{8} + \frac{5}{16}\right)B_4\right],$$

$$\int_0^1 \frac{B_x(\xi)\xi^2\mathrm{d}\xi}{x-\xi} = \int_0^1 \frac{B_1\xi^2 + B_2\xi^3 + B_3\xi^4 + B_4\xi^5}{\sqrt{\xi(1-\xi)}(x-\xi)}\mathrm{d}\xi$$
$$= -\pi\left[\left(x + \frac{1}{2}\right)B_1 + \left(x^2 + \frac{x}{2} + \frac{3}{8}\right)B_2 + \left(x^3 + \frac{x^2}{2} + \frac{3x}{8} + \frac{5}{16}\right)\right.$$
$$\left.\times B_3 + \left(x^4 + \frac{x^3}{2} + \frac{3x^2}{8} + \frac{5x}{16} + \frac{35}{128}\right)B_4\right],$$

$$\int_0^1 B_x(\xi)\mathrm{d}\xi = \int \frac{B_1 + B_2\xi + B_3\xi^2 + B_4\xi^3}{\sqrt{\xi(1-\xi)}}\mathrm{d}\xi = \pi\left(B_1 + \frac{B_2}{2} + \frac{3B_3}{8} + \frac{5B_4}{16}\right).$$

Since identical formulae hold for B_y except for substitution of B_i with C_i, we can write Equations (3.5) as

$$
\begin{aligned}
&\left(-A_2x-2A_3x^2\right)\left[C_2+\left(x+\frac{1}{2}\right)C_3+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)C_4\right]+(A_2+A_3x)\\
&\quad\times\left[C_1+\left(x+\frac{1}{2}\right)C_2+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)C_3+\left(x^3+\frac{x^2}{2}+\frac{3x}{8}+\frac{5}{16}\right)C_4\right]\\
&\quad+A_3\left[\left(x+\frac{1}{2}\right)C_1+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)C_2+\left(x^3+\frac{x^2}{2}+\frac{3x}{8}+\frac{5}{16}\right)\right.\\
&\quad\left.\times C_3+\left(x^4+\frac{x^3}{2}+\frac{3x^2}{8}+\frac{5x}{16}+\frac{35}{128}\right)C_4\right]\\
&\quad+\left[B_2+\left(x+\frac{1}{2}\right)B_3\right]=\frac{T_x}{\pi}=-\frac{\sigma_x^\infty}{\pi}\left(A_1+2A_2x+3A_3x^2\right)+\frac{\tau_{xy}^\infty}{\pi},
\end{aligned}
\tag{4.2a}
$$

$$
\begin{aligned}
&\left(-A_2x-2A_3x^2\right)\left[B_2+\left(x+\frac{1}{2}\right)B_3+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)B_4\right]+(A_2+A_3x)\\
&\quad\times\left[B_1+\left(x+\frac{1}{2}\right)B_2+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)B_3+\left(x^3+\frac{x^2}{2}+\frac{3x}{8}+\frac{5}{16}\right)B_4\right]\\
&\quad+A_3\left[\left(x+\frac{1}{2}\right)B_1+\left(x^2+\frac{x}{2}+\frac{3}{8}\right)B_2+\left(x^3+\frac{x^2}{2}+\frac{3x}{8}+\frac{5}{16}\right)\right.\\
&\quad\left.\times B_3+\left(x^4+\frac{x^3}{2}+\frac{3x^2}{8}+\frac{5x}{16}+\frac{35}{128}\right)B_4\right]\\
&\quad+\left[C_2+\left(x+\frac{1}{2}\right)C_3\right]=\frac{T_y}{\pi}=-\frac{\tau_{xy}^\infty}{\pi}\left(A_1+2A_2x+3A_3x^2\right)+\frac{\sigma_y^\infty}{\pi}.
\end{aligned}
\tag{4.2b}
$$

Equation (3.6) yield

$$
B_1+\frac{B_2}{2}+\frac{3B_3}{8}+\frac{5B_4}{16}=0,\quad C_1+\frac{C_2}{2}+\frac{3C_3}{8}+\frac{5C_4}{16}=0.
\tag{4.3}
$$

Frobenius' method consists of equating terms of the same order in x in (4.2). This, together with (4.3), generates a system of linear equations in B_i, C_i, which can be prolonged indefinitely by taking more and more coefficients in (4.1). If we content ourselves to arrest Frobenius' procedure to terms of first order in x, it is sufficient to retain the coefficients $B_1, B_2, B_3, C_1, C_2, C_3$ in the expansions (4.1). For this level of approximation, (4.3) reduces to

$$
B_1+\frac{B_2}{2}+\frac{3B_3}{8}=0,\quad C_1+\frac{C_2}{2}+\frac{3C_3}{8}=0,
\tag{4.4}
$$

and (4.2) yields the equations

$$B_2+\frac{B_3}{2}+\left(A_2+\frac{A_3}{2}\right)C_1+\left(\frac{A_2}{2}+\frac{3A_3}{8}\right)C_2+\left(\frac{3A_2}{8}+\frac{5A_3}{16}\right)C_3$$
$$=\frac{1}{\pi}\left(\tau_{xy}^{\infty}-A_1\sigma_x^{\infty}\right), \tag{4.5a}$$
$$C_2+\frac{C_3}{2}+\left(A_2+\frac{A_3}{2}\right)B_1+\left(\frac{A_2}{2}+\frac{3A_3}{8}\right)B_2+\left(\frac{3A_2}{8}+\frac{5A_3}{16}\right)B_3$$
$$=\frac{1}{\pi}\left(\sigma_y^{\infty}-A_1\tau_{xy}^{\infty}\right), \tag{4.5b}$$
$$B_3+A_3C_2+\frac{3A_3C_3}{4}+2A_3C_1=-\frac{2}{\pi}A_2\sigma_x^{\infty}, \tag{4.5c}$$
$$C_3+A_3B_2+\frac{3A_3B_3}{4}+2A_3B_1=-\frac{2}{\pi}A_2\tau_{xy}^{\infty}. \tag{4.5d}$$

Solving the system obtained by combining (4.5) with (4.3) (with $B_4=C_4=0$) and neglecting terms of second order in A_1, A_2, A_3 we obtain

$$\pi B_1=-\frac{\tau_{xy}^{\infty}}{2}+\sigma_x^{\infty}\left(\frac{A_1}{2}+\frac{A_2}{4}\right)+\sigma_y^{\infty}\frac{A_3}{16}, \tag{4.6a}$$
$$\pi B_2=\tau_{xy}^{\infty}+\sigma_x^{\infty}(A_2-A_1)-\sigma_y^{\infty}\frac{A_3}{8}, \tag{4.6b}$$
$$\pi B_3=-2A_2\sigma_x^{\infty}, \tag{4.6c}$$
$$\pi C_1=-\frac{\sigma_y^{\infty}}{2}+\tau_{xy}^{\infty}\left(\frac{A_1}{2}+\frac{A_2}{4}+\frac{A_3}{16}\right), \tag{4.6d}$$
$$\pi C_2=\sigma_y^{\infty}+\tau_{xy}^{\infty}(A_2-A_1)-\tau_{xy}^{\infty}\frac{A_3}{8}, \tag{4.6e}$$
$$\pi C_3=-2A_2\tau_{xy}^{\infty}. \tag{4.6f}$$

The arrest of Frobenius' method to the first three terms in (4.1) may appear a gross approximation, but the following illustrative examples show that this is not the case.

5. Evaluation of the stress intensity factors of illustrative curved cracks

Having found the coefficients $B_1, B_2, \ldots, C_3$ we can calculate the stress components produced by the presence of the curved crack, the crack opening displacements, and in particular, the stress intensity factors. The stress intensity factors can be readily evaluated from the near-tip behavior of the dislocation densities referred to the local normal and tangent to the crack surfaces, B_r, B_θ, which are related to the previously defined Cartesian components by

$$B_r=B_x+B_y\sin\theta\cong B_x+\theta B_y, \tag{5.1a}$$
$$B_\theta=-B_x\sin\theta+B_y\cong -B_x\theta+B_y, \tag{5.1b}$$

where the last terms on the right-hand side of (5.1) reflect the linearization consistent with the assumptions previously stated about the shape of the crack. The mode-I and mode-II stress intensity factors are given, respectively, by

$$K_{II} = \sqrt{2\pi s}\pi B_r, \tag{5.2a}$$

$$K_I = \sqrt{2\pi s}\pi B_\theta. \tag{5.2b}$$

Combining (5.2) with (5.1), and noting that $\theta \cong y' = A_1 + 2A_2x$, the stress intensity factors at the left crack tip ($x = 0$) reduce to

$$K_I = -\pi\,(C_1 - A_1B_1)\,, \tag{5.3a}$$

$$K_{II} = -\pi\,(B_1 + A_1C_1)\,. \tag{5.3b}$$

Substitution of (4.6a), (4.6b), (4.6c) and (4.6d) into (5.3), and normalizing with respect to the half-crack length, 1/2, we obtain

$$\overline{K_I} \equiv \frac{K_I}{\sqrt{\pi/2}} = \sigma_y^\infty - \tau_{xy}^\infty\left(2A_1 + \frac{A_2}{2} + \frac{A_3}{8}\right), \tag{5.4a}$$

$$\overline{K_{II}} \equiv \frac{K_{II}}{\sqrt{\pi/2}} = \tau_{xy}^\infty - \sigma_x^\infty\left(A_1 + \frac{A_2}{2}\right) - \sigma_y^\infty\left(-A_1 + \frac{A_3}{8}\right). \tag{5.4b}$$

We now assess Frobenius' method by comparing the stress intensity factors given by (5.4) with those of curved cracks treated by Chen et al. (1991), Brandinelli (1997) and Cotterell and Rice (1980). We first consider a parabolic crack defined by the equation $y(x) = 2\alpha x(1-x)$ in a plate subjected to the fundamental state, $\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty$. In terms of the geometric parameter α, the coefficients in (2.1) have the values $A_1 = 2\alpha$, $A_2 = -2\alpha$, $A_3 = 0$. The stress intensity factors, given by

$$\overline{K_I} = \sigma_y^\infty - 3\alpha\tau_{xy}^\infty, \tag{5.5a}$$

$$\overline{K_{II}} = \tau_{xy}^\infty + 2\alpha\sigma_{yy}^\infty - \alpha\sigma_x^\infty, \tag{5.5b}$$

are the same as those derived by Cotterell and Rice (1980), using a rigorous perturbation technique, for a circular arc shaped crack.

In order to illustrate the precision of Frobenius' method we compare the results of formulae (5.5a) and (5.5b) with those obtained by Chen et al. (1991) for the problem of a parabolic crack of length 2 and height α subject to uniform stresses at infinity (Figure 2a). The authors drew the graphs of the normalized stress intensity factors as functions of α considering separate action of the far-field stresses $\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty$. The corresponding curves are indicated by $\overline{K_{II}}(\sigma_x^\infty)$, $\overline{K_{II}}(\sigma_y^\infty)$, $\overline{K_I}(\tau_{xy}^\infty)$ in Figure 2b. Formulae (5.5a) and (5.5b) are represented by the straight dashed lines in Figure 2b, which are tangent to the numerical results at the origin, and provide satisfactory stress intensity factor approximations for small values of α.

Let us instead consider the cubic parabola $y(x) = \alpha x(x-1)^2$ having a slope 2α at the left tip and flat at the other tip. The coefficients in (2.1) are now $A_1 = A_3 = 2\alpha$, $A_2 = -4\alpha$, and (5.4) give

$$\overline{K_I} = \sigma_y^\infty - 2.25\alpha\tau_{xy}^\infty, \tag{5.6a}$$

$$\overline{K_{II}} = \tau_{xy}^\infty + 1.75\alpha\sigma_y^\infty. \tag{5.6b}$$

If, instead, the cubic parabola is flat at the origin and inclined at its right tip, its equation is $y(x) = 2\alpha x^2(1-x)$, and $A_1 = 0$, $A_2 = -A_3 = 2\alpha$. Thus (5.4) give

$$\overline{K_I} = \sigma_y^\infty - 0.75\alpha\tau_{xy}^\infty, \tag{5.7a}$$

$$\overline{K_{II}} = \tau_{xy}^\infty - \alpha\sigma_x^\infty + 0.25\alpha\sigma_y^\infty. \tag{5.7b}$$

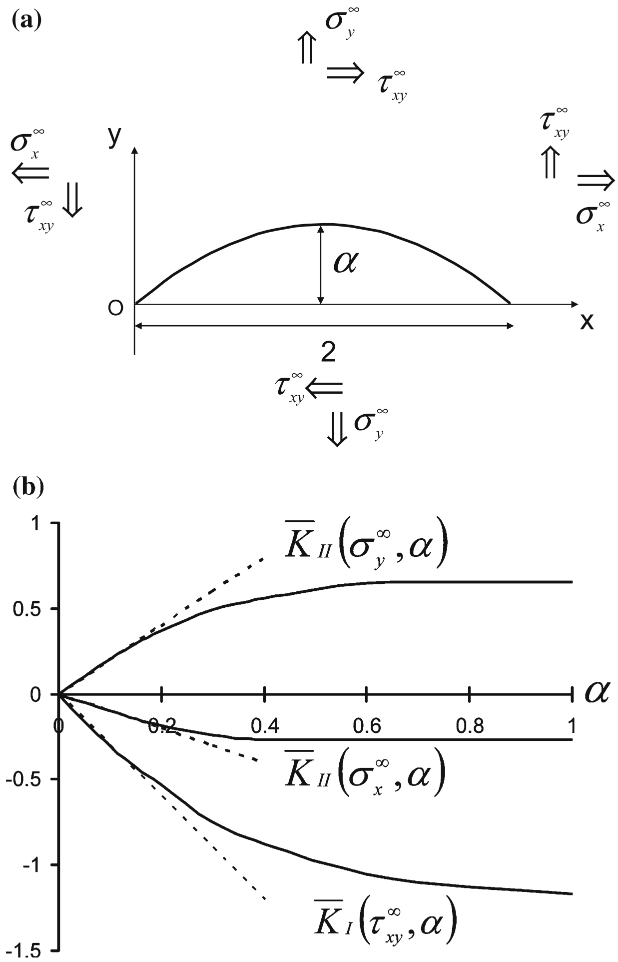

Figure 2. Parabolic crack subjected to uniform far-field stresses; (a) geometry of crack; (b) normalized stress intensity factors.

For sufficiently small values of $\alpha (\alpha \leqslant 0.2)$ the numerical values of the stress intensity factors furnished by formulas (5.5)–(5.7) differ less than 20% from those obtained by Chen et al. (1991). This case can also be geometrically illustrated by a graph similar to that sketched in Figure 2b.

6. Extension to elastically cohesive cracks

Frobenius' method can be extended without difficulty to the case in which a cohesive force hinders the free relative opening and sliding of the crack surfaces. Let us assume that the cohesive force is proportional to the crack opening and crack sliding displacements, and denote the elastic coefficient of the cohesion by k. This implies that the surfaces of the crack are no longer traction free, but subjected to restoring forces with components

$$R_x = k[u_x], \quad R_y = k[u_y], \tag{6.1}$$

where $[u_x], [u_y]$ are the previously defined displacement jumps, which by definition are related to the dislocation densities by the formulae

$$[u_x] = \int_0^x B_x(\xi)\, \mathrm{d}\xi, \quad [u_y] = \int_0^x B_y(\xi)\, \mathrm{d}\xi. \tag{6.2}$$

The total stress intensity factor is equal to the linear superposition of the stress intensity factor due to the applied stresses and the (negative) stress intensity factor contributed by the cohesive forces.

Recalling (4.1) and performing the integration in (6.2), we obtain

$$[u_x] = \int_0^x \frac{B_1 + \xi B_2 + \xi^2 B_3 + \cdots}{\sqrt{\xi(1-\xi)}}\, \mathrm{d}\xi = 2B_1 \arcsin\sqrt{x} + B_2\left(\arcsin\sqrt{x} - \sqrt{x(1-x)}\right)$$
$$+ B_3\left(\frac{3}{4}\arcsin\sqrt{x} - \frac{3}{4}\sqrt{x(1-x)} - \frac{1}{2}x\sqrt{x(1-x)}\right) + \cdots, \tag{6.3}$$

and an identical expression for $[u_y]$ except for substitution of B_i by C_i.

Applying the expansions

$$\arcsin\sqrt{x} = \sqrt{x} + \frac{1}{6}\left(\sqrt{x}\right)^3 + \frac{3}{40}\left(\sqrt{x}\right)^5 + \cdots,$$

$$\sqrt{x(1-x)} = \sqrt{x} - \frac{1}{2}\left(\sqrt{x}\right)^3 - \frac{1}{8}\left(\sqrt{x}\right)^5 + \cdots,$$

allows us to write (6.3) to the third order term in $\sqrt{x}$ as

$$[u_x] = 2B_1\sqrt{x} + \left(\frac{2B_1}{3} - \frac{B_2}{3} - \frac{3B_3}{4}\right)\left(\sqrt{x}\right)^3 + \cdots, \tag{6.4}$$

with the same expression holding for $[u_y]$ save replacement of B_i by C_i.

For this cohesive crack equations (2.6) must be modified into

$$\left(\sigma_x^\infty + \sigma_x\right) n_x + \left(\tau_{xy}^\infty + \tau_{xy}\right) n_y - k[u_x] = 0, \tag{6.5a}$$

$$\left(\tau_{xy}^\infty + \tau_{xy}\right) n_x + \left(\sigma_y^\infty + \sigma_y\right) n_y - k[u_y] = 0. \tag{6.5b}$$

Substitution of (6.4) into (6.5), retaining terms of order zero and one in $\sqrt{x}$, and solving the system of equations, we recover Equations (4.4) and (4.5a), (4.5b), while, instead of (4.5c), (4.5d) we obtain $B_1 = C_1 = 0$. According to (5.3), the total stress intensity factor is therefore equal to zero, independent of the value of k. We note that the total stress intensity factor could not be zero for a cohesive law associated with zero force at zero crack opening displacement. However, the approximation produced by Frobenius' method is qualitatively consistent with the asymptotic limit derived by Rose (1987) for the total stress intensity factor of cracks of length $2a$ bridged by 'stiff' springs (or long cracks)

$$\overline{K_i} \equiv \frac{K_i}{\sqrt{\pi a}} \approx \frac{\sigma_{ij}^\infty}{\sqrt{k}}, \quad ka \gg 1. \tag{6.6}$$

The mathematical result that the presence of a relatively soft adhesive between the faces of a sufficiently long crack is sufficient to annihilate the stress singularities may appear surprising. Apparently, Venetian carpenters' practice (since the Middle-Age) of inserting bitumen between the surfaces of cracks in the keels not only water-sealed the 'Repubblica's' galleys, but also mitigated the propagation of fractures.

References

Brandinelli, L. (1997). Characterization of planar curvilinear cracks and development of a probabilistic crack propagation model for brittle materials. M.S. Thesis, Case Western Reserve University, Cleveland, OH.

Chen, Y.Z., Gross, D. and Huang, Y.J. (1991). Numerical solution of the curved crack problem by means of polynomial approximation of the disclocation distribution. *Engineering Fracture Mechanics* **39**(5), 791–797.

Cotterell, B. and Rice, J.R. (1980). Slightly curved or kinked cracks. *International Journal of Fracture* **16**(2), 155–169.

Dreilich, L. and Gross, D. (1985). Der gekrümmte Riss. *Zeitschrift Fur Angewandte Mathematik und Mechanik* **65**, 132–134.

Dundurs, J. and Sendeckyj, G.P. (1965). Behavior of an edge dislocation near a bimaterial interface. *Journal of Applied Physics* **36**(10), 3353–3354.

Grigolyuk, E. and Tolkachev, V. (1987). *Contact Problems in the Theory of Plates and Shells*. Mir, Moscow.

Muskhelishvili, N.I. (1953). *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Groningen.

Rose, L.R.F. (1987). Crack reinforcement by distributed springs. *Journal of the Mechanics and Physics of Solids* **35**(4), 383–405.

Savruk, M.P. (1981). *Two-Dimensional Problems of Elasticity for Body with Crack*. The University Press, Kiev (in Russian).