Analysis of Branched Interface Cracks

A solution is presented for the problem of a finite length crack branching off the interface between two bonded dissimilar isotropic materials. Results are presented in terms of the ratio of the energy release rate of a branched interface crack to the energy release rate of a straight interface crack with the same total length. It is found that this ratio reaches a maximum when the interface crack branches into the softer material. Longer branches tend to have smaller maximum energy release rate ratio angles indicating that all else being equal, a branch crack will tend to turn back parallel to the interface as it grows.

1 Introduction

The interface crack problem was addressed in the late 1950's by Williams (1959). Williams used an eigenfunction expansion technique to solve the interface crack problem and discovered complex singularities and rapid oscillating stresses near the crack tip. Williams also estimated the oscillatory region to be very small.

In the mid 1960's, various workers (England (1965), Erdogan (1965), Rice and Sih (1965)) came up with closed-form solutions to the interface problem. These solutions verified Williams' (1959) finding of complex singularities and rapidly oscillating stresses near the crack tip. England (1965) also pointed out that the complex singularities lead to the physical impossibility of the crack faces interpenetrating near the crack tips. Like Williams (1959), these workers found the region where these phenomena occur was very small.

In the mid-1970's Comninou (1977) put forth an alternate solution that addressed the difficulties at the crack tip. Comninou resolved the problems at the crack tip by solving the interface crack problem assuming there is a small contact zone near the crack tips. This zone was found to be extremely small. While calculations were carried out for a limited range of Dundurs' constants (Dundurs 1969), Comninou's work provides a rational explanation of the oddities occurring near the crack tips.

Lately, the interface crack problem has received renewed attention. Gautesen and Dundurs (1987) have solved the problem formulated by Comninou (1977) using quickly convergent series. Unlike Comninou's (Comninou (1977) results which were carried out for only severe material mismatches, Gautesen and Dundurs obtain results covering all material combinations. Symington (1987) has completed Williams' (Williams 1959) eigenfunction analysis by finding a set of integer eigenvalues left out by Williams. Rice (1988) has proposed that the peculiarities at the crack tip are not too critical. In an argument similar to the concept of small-scale yielding, he asserts that the complex stress intensity factors will be indicative of the general state of the crack tip even if they do not correctly represent the state immediately surrounding the crack tip. Rice also provides a framework to interpret the complex intensity factor in terms of the classical form stress intensity factors $K_I$ and $K_{II}$. He also includes an interesting discussion of the interaction of the crack length and load phase angle influence. Hutchinson, Mear, and Rice (1987) have looked at a crack completely in one body but very near the interface. Their solution, based on dimensional arguments, energy considerations, and the representation of the crack as a distribution of dislocations, examines criteria for crack growth parallel to the material interface. Park and Earmme (1986) use various integrals to describe the interface crack tip characteristics and include a discussion of the properties of these integrals. Sih and Asaro (1988) have investigated the interface crack with plasticity. They find that the pathological features of the linear elastic analyses are lessened by nonlinear behavior and that the linear elastic solutions are good in regions not immediately around the crack tip. Delale and Erdogan (1988) have looked at the interface crack by modeling the interface as a transition material; they put the crack in a thin third layer between the two half-spaces. They present some results regarding the direction of maximum $K_I$ of the crack. Generally, $K_I$ is larger as the crack tends toward the softer material. He and Hutchinson (1988) have performed an analysis very similar to the present one for the case of a branched semi-infinite crack.
They have presented extensive results on the initial branching angle, couched in terms of strain energy release rates.

In this paper, the branching of a finite length crack is analyzed. For small branch lengths the asymptotic results of He and Hutchinson are reproduced. For longer branch lengths the behavior changes as the global portion of the stress field influences further branch growth. Based on the obtained results, some conjectures are made concerning the fate of branching cracks.

2 Formulation

The problem configuration is depicted in Fig. 1, which shows the main crack, the branch crack, the loading at infinity, and the coordinate system. In the upper half-plane ($S_1$), the shear modulus is $\mu_1$ and Poisson's ratio is $\nu_1$. In the lower half-plane ($S_2$), the shear modulus is $\mu_2$ and Poisson's ratio is $\nu_2$. The boundary conditions for this problem require that both the main and branch cracks are traction-free and that at infinity the stresses approach zero. Single-valuedness of the displacements at infinity is also required.

Some techniques useful for dealing with interface boundary conditions which are used in the problem formulation are presented first. These results follow directly from the formulation used by Clements (1971) to treat interface cracks in anisotropic materials.

In terms of the Muskhelishvili (1953) potentials, the stresses and displacements in isotropic elastic bodies may be expressed as follows:

\begin{align}
\sigma_{yy} - ir_{yy} &= \Phi_1(z) + z\Phi_1'(z) + \Psi_1(z) \quad (1) \\
\sigma_{yy} - i\sigma_{xy} &= 2[\Phi_2(z) + \Phi_3(z)] \quad (2) \\
2\mu_1\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}ight) &= \kappa \Phi_1(z) - [\Phi_1(z) + z\Phi_1'(z) + \Psi_1(z)] \quad (3)
\end{align}

in which the subscript $i (i = 1, 2)$ denotes “in region $i$”, $\Phi_1, \Psi_1$ correspond to the potentials for the upper half-plane and $\Phi_2, \Psi_2$ correspond to the potentials for the lower half-plane. Moreover, $z$ is the complex variable $x + iy$, the prime denotes differentiation with respect to $z$, an overbar denotes conjugation, $\kappa = 3 - 4\nu$ for plane strain, and $\kappa = \frac{3 - \nu}{1 + \nu}$ for plane stress.

For interface problems, it is convenient to introduce additional potentials as follows. Making use of the fact that if $f(z)$ is analytic for $z$ in region $R$, then $f(z) = \overline{f(z)}$ is analytic for $\overline{z}$ in region $R$, the following analytic jump potentials are constructed:

\begin{align}
\Omega_1 &= \left\{ \begin{array}{ll}
\Phi_1(z) - \overline{\Phi_1(z)} + z\Phi_1'(z) + \Psi_1(z) & \text{on } S_1 \\
\Phi_2(z) - \overline{\Phi_1(z)} + z\Phi_2'(z) + \Psi_2(z) & \text{on } S_2
\end{array} \right. \quad (4a) \\
\Omega_2 &= \left\{ \begin{array}{ll}
\frac{k_1}{2\mu_1} \Phi_1(z) + \frac{1}{2\mu_2} \overline{\Phi_2(z)} + z\Phi_1'(z) + \Psi_1(z) & \text{on } S_1 \\
\frac{k_2}{2\mu_2} \Phi_2(z) + \frac{1}{2\mu_1} \overline{\Phi_1(z)} + z\Phi_2'(z) + \Psi_2(z) & \text{on } S_2
\end{array} \right. \quad (4b)
\end{align}

Note that in terms of these jump potentials, the interface conditions can be expressed as:

\begin{align}
\left(\sigma_{yy} - ir_{yy}\right)z - \left(\sigma_{yy} - ir_{yy}\right)\overline{z} &= \Omega_1 S_1(z) - \Omega_1 S_2(z) \quad (5a) \\
\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\overline{z} &= \Omega_2 S_2(z) - \Omega_2 S_1(z) \quad (5b)
\end{align}

The $\Omega$ potentials thus represent the jump in traction ($\Omega_1$) and displacement ($\Omega_2$) across the interface. The inverse relationship between the jump and Muskhelishvili potentials is determined algebraically from equations (4):

\begin{align}
\Phi_1(z) &= Q_1 \left[ \frac{1}{2\mu_2} \Omega_1 S_1(z) + \Omega_1 D_1(z) \right] \quad (6a) \\
\Psi_1(z) &= Q_2 \left[ \frac{1}{2\mu_1} \Omega_1 S_2(z) + \Omega_1 D_2(z) \right] \quad (6b) \\
\Phi_2(z) &= Q_1 \left[ \frac{1}{2\mu_2} \Omega_2 S_1(z) + \Omega_2 D_1(z) \right] \quad (6c) \\
\Psi_2(z) &= Q_2 \left[ \frac{1}{2\mu_1} \Omega_2 S_2(z) + \Omega_2 D_2(z) \right] \quad (6d)
\end{align}

where

\begin{align}
Q_1 &= \frac{2\mu_2}{\mu_1 + k_1\mu_2} \quad (7a) \\
Q_2 &= \frac{2\mu_2}{\mu_2 + k_1\mu_1} \quad (7b)
\end{align}

This formulation allows for straightforward development of the various interaction potentials derived below.

3 Solution

The solution is obtained by a Green's function technique based on distributing dislocation singularities along the branch. The solution for the interaction between an interface crack and a dislocation is found by superposing the solutions for: (1) a dislocation in $S_1$ near an interface; (2) two perfectly bonded semi-infinite bodies loaded at infinity with $\sigma_{yy}$ and $r_{yy}$; and (3) an interface crack loaded with the negative of the stresses produced by (1) and (2). This solution for a single dislocation interacting with an interface crack (with the appropriate far-field loading) is then modified to model a distribution of dislocations along the branch, and by requiring zero tractions along the branch face a singular integral equation is obtained for the unknown dislocation distribution. The relevant physical quantities can be calculated after numerically solving the integral equation.

Most of the component solutions to be superposed are well-known. Nevertheless, it is convenient to rewrite them here using the framework presented in Section 2. This will provide a consistent formulation, as well as a demonstration of the generality of the approach.

Consider first a dislocation near an interface (see Head,
1953; Dundurs and Sendeckyj (1965) located at a point \( z_0 \) in the upper half-plane. The continuity of tractions and displacements across the interface in combination with equation (5) implies the following boundary conditions on the jump potentials:

\[
\begin{align*}
\Omega_{11}^0 - \Omega_{12}^0 &= 0, \\
\Omega_{21}^0 - \Omega_{22}^0 &= 0
\end{align*}
\]

The singular portion of the jump potentials can be obtained directly by transforming the known singular potentials for a dislocation in a homogeneous full-plane into jump potentials. For a full-plane dislocation in region 1 we have:

\[
\begin{align*}
\Phi_1^0, \text{ Singular} &= \frac{A}{z-z_0}, \\
\Phi_2^0, \text{ Singular} &= \frac{A}{z-z_0} - \frac{z_0-z_0}{(z-z_0)^2}, \\
\Psi_1^0, \text{ Singular} &= \frac{A}{z-z_0} + A \frac{z_0-z_0}{(z-z_0)^2}, \\
\Psi_2^0, \text{ Singular} &= 0
\end{align*}
\]

in which \( A = \mu_1 \mathbf{r}^{(1)} [u_x] + i \mathbf{r}^{(1)} [v_y] \) and \( [u_x] \) and \( [v_y] \) representing the jumps in the tangential and normal displacements accrued upon circling the dislocation. The corresponding \( \Omega \) potentials are determined from equation (4):

\[
\begin{align*}
\Omega_{11}^0, \text{ Singular} &= \frac{A}{z-z_0}, \\
\Omega_{12}^0, \text{ Singular} &= \frac{A}{z-z_0} - \frac{z_0-z_0}{(z-z_0)^2}, \\
\Omega_{21}^0, \text{ Singular} &= \frac{\kappa_1}{2 \mu_1} A, \\
\Omega_{22}^0, \text{ Singular} &= \frac{\kappa_1}{2 \mu_1} A + \frac{z_0-z_0}{(z-z_0)^2}
\end{align*}
\]

Obviously, the boundary conditions given by equation (8) are not satisfied. To patch up the interface boundary conditions, potentials of the following form are added, which are analytic in their respective regions:

\[
\begin{align*}
\Phi_1^0, \text{ Continuation} &= \Phi_1^0, \text{ Singular}; \ z \in S_1, \\
\Phi_2^0, \text{ Continuation} &= \Phi_2^0, \text{ Singular}; \ z \in S_2
\end{align*}
\]

with similar expressions for the displacement jump potentials. Using equation (6) to invert these to regular potentials gives the additional terms to be added to the singular potentials in equation (9) to give the desired solution (i.e., \( \Phi^0 = \Phi^0, \text{ Continuation} + \Phi^0, \text{ Singular} \), etc.):

\[
\begin{align*}
\Phi_1^c &= A \left( \frac{(\alpha - \beta)}{(1 + \beta)(z-z_0)} \right) + A \left( \frac{(\alpha - \beta)}{(1 + \beta)(z-z_0)^2} \right), \\
\Phi_2^c &= A \left( \frac{(\alpha - \beta)}{(1 + \beta)(z-z_0)} \right) - A \left( \frac{(\alpha - \beta)}{(1 + \beta)(z-z_0)^2} \right), \\
\Psi_1^c &= A \left( \frac{(\alpha + \beta)}{(1 - \beta)(z-z_0)} \right), \\
\Psi_2^c &= A \left( \frac{(\alpha + \beta)}{(1 - \beta)(z-z_0)} \right) - 2 \Psi_1^c, \text{ Continuation}
\end{align*}
\]

where the \( \Phi \) and \( \Psi \) potentials are as follows:

\[
\begin{align*}
\Phi_1^c &= \Phi_1^0, \text{ Continuation} - 2 \Phi_1^c, \text{ Continuation}, \\
\Phi_2^c &= \Phi_2^0, \text{ Continuation} - 2 \Phi_2^c, \text{ Continuation}
\end{align*}
\]

The constant potentials corresponding to the far-field loading can be derived from the jump potentials by simply starting with constant \( \Omega \)'s, which automatically satisfy the interface conditions, and inverting to determine the corresponding \( \Phi \)'s. The details of this operation are in the Appendix; the results are as follows:

\[
\Phi_1^0 = \frac{\alpha_m^0 + \alpha_{\text{ext}}^0}{4} \left( \frac{1 - \alpha}{2} \right) + \left( \frac{\alpha + \beta}{2} \right) \frac{r_y^m}{2} + \frac{1 - \alpha}{2} IC_S
\]

\[
\Phi_2^0 = \frac{\alpha_m^0 - \alpha_{\text{ext}}^0}{4} \left( \frac{1 + \alpha}{2} \right) + \left( \frac{\alpha + \beta}{2} \right) \frac{r_y^m}{2} + \frac{1 - \alpha}{2} IC_S
\]

\[
\Psi_1^0 = \frac{\alpha_m^0 + \alpha_{\text{ext}}^0}{4} \left( \frac{1 - \alpha}{2} \right) + \left( \frac{\alpha + \beta}{2} \right) \frac{r_y^m}{2} + \frac{1 - \alpha}{2} IC_S
\]

\[
\Psi_2^0 = \frac{\alpha_m^0 - \alpha_{\text{ext}}^0}{4} \left( \frac{1 + \alpha}{2} \right) + \left( \frac{\alpha + \beta}{2} \right) \frac{r_y^m}{2} + \frac{1 - \alpha}{2} IC_S
\]

\[
\Phi_1^c = \Phi_1^0, \text{ Continuation} - 2 \Phi_1^c, \text{ Continuation}, \\
\Phi_2^c = \Phi_2^0, \text{ Continuation} - 2 \Phi_2^c, \text{ Continuation}
\]

\[
\Psi_1^c = \Phi_1^0, \text{ Continuation} + \Phi_1^c, \text{ Continuation}, \\
\Psi_2^c = \Phi_2^0, \text{ Continuation} + \Phi_2^c, \text{ Continuation}
\]

The details of this operation are in the Appendix; the results are as follows:

\[
\begin{align*}
\Phi_1^c &= \Phi_1^0, \text{ Continuation} - 2 \Phi_1^c, \text{ Continuation}, \\
\Phi_2^c &= \Phi_2^0, \text{ Continuation} - 2 \Phi_2^c, \text{ Continuation}
\end{align*}
\]

\[
\begin{align*}
\Psi_1^c &= \Phi_1^0, \text{ Continuation} + \Phi_1^c, \text{ Continuation}, \\
\Psi_2^c &= \Phi_2^0, \text{ Continuation} + \Phi_2^c, \text{ Continuation}
\end{align*}
\]

In determining the relations (14), it turns out that the \( x \)-direction stresses are related:

\[
\sigma_{xx} = \frac{4 \beta - 2 \alpha}{1 - \alpha} \sigma_{x\text{ext}}^0 + \frac{1 + \alpha}{1 - \alpha} \sigma_{x\text{ext}}^1
\]
To obtain these potentials, first consider the interface crack under arbitrary crack face loading (the standard solution has been presented by many investigators e.g., England, 1965; Erdogan, 1965; Rice and Sih, 1965). For a crack on an interface, the interface boundary conditions are:

\[
\begin{align*}
\sigma_{yy} - i \tau_{yx} &= 0 \quad |x| < \infty, \quad y = 0 \\
(u' - iv')_1 - (u' - iv')_2 &= 0 \quad |x| > c
\end{align*}
\]

(17a)

(17b)

These boundary conditions, in terms of the standard potentials, are somewhat cumbersome. However, as shown by equation (6), in terms of the jump potentials the first boundary condition is simply:

\[
\Omega_{Dd}(x) - \Omega_{Dd}(x) = 0 \quad |x| < \infty. \quad (18)
\]

Using an argument similar to the one in the far-field solution (see the Appendix), since \(\Omega_s\) is analytic everywhere and bounded, by Liouville's theorem it must be constant. Moreover, for zero stress at infinity, \(\Omega_s = 0\).

The second boundary condition in terms of the jump potentials is:

\[
\Omega_{Dd}(x) - \Omega_{Dd}(x) = 0 \quad |x| > c. \quad (19)
\]

The third boundary condition in terms of the jump potentials is:

\[
Q_1 \Omega_{Dd}(x) + Q_2 \Omega_{Dd}(x) = f(x) \quad |x| < c \quad (20a)
\]

or

\[
\Omega_{Dd} + m \Omega_{Dd} = \frac{1}{Q_1} f(x) \quad |x| < c \quad (21a)
\]

\[
m = \frac{Q_2}{Q_1} = \frac{1 + \beta}{1 - \beta}. \quad (21b)
\]

These last two boundary conditions (equations (19) and (21a)) define a Hilbert problem, with the well-known solution:

\[
\Omega_D(z) = \frac{X(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{Q_1 X^+(x)(x-z)} \, dx + X(z) P(z) \quad (22)
\]

in which

\[
X(z) = (z-c)^{-1} \gamma^{-1} (z+c)^{-1}
\]

\[
\gamma = \frac{1}{2} - \frac{i}{2\pi} \log m = \frac{1}{2} + i\epsilon
\]

\[
\epsilon = \frac{1}{2\pi} \log \frac{1}{m}
\]

and \(P(z)\) is a polynomial determined by far-field behavior. After \(\Omega_D\) is obtained, the jump potentials can be inverted back to standard potentials using equation (6).

To remove the stresses on the crack caused by the dislocation solution, the integration in equation (22) is carried out with \(f(x)\) equal to the opposite of the tractions due to a dislocation near the interface. These tractions are obtained by substituting equations (9) and (12) into equation (1):

\[
\sigma_{yx} - i \tau_{xy} = A \left[ \frac{1 + \alpha}{1 - \beta} \left( \frac{1}{x - z_0} \right) + \frac{1 + \alpha}{1 + \beta} \left( \frac{1}{x - z_0} \right) \right]
\]

\[
+ A \left[ \frac{1 + \alpha}{1 + \beta} \left( \frac{z_0 - \bar{z}}{x - z_0} \right) + \frac{1 + \alpha}{1 + \beta} \left( \frac{z_0 - \bar{z}}{x - z_0} \right) \right]. \quad (23)
\]

The resulting jump potentials are

\[
\Omega_{Dislocation}^C = 0 \quad (24a)
\]

\[
\Omega_{Dd}, \text{ Dislocation} = -\frac{2A}{1 + m} \left[ \frac{1 + \alpha}{1 - \beta} F(z, z_0) + \frac{1 + \alpha}{1 + \beta} F(z, \bar{z}_0) \right]
\]

\[
- \frac{2A}{1 + m} \left[ \frac{1 + \alpha}{1 - \beta} G(z, z_0) + \frac{1 + \alpha}{1 + \beta} G(z, \bar{z}_0) \right]
\]

\[
+ X(z) P(z) \quad (24b)
\]

in which

\[
F(z, a) = \frac{1}{2} \left[ 1 - \frac{X(z)}{X(a)} \right] \quad (25a)
\]

\[
G(z, a) = \frac{\partial F(z, a)}{\partial a}. \quad (25b)
\]

Inverting these to standard potentials gives:

\[
\Phi_{Dd}, \text{ Dislocation} = -\frac{2A}{1 + m} \left[ \frac{1 + \alpha}{1 - \beta} F(z, z_0) \right]
\]

\[
+ \frac{1 + \alpha}{1 + \beta} F(z, \bar{z}_0) \quad (26a)
\]

\[
- \frac{2A}{1 + m} \left[ \frac{1 + \alpha}{1 - \beta} G(z, z_0) \right]
\]

\[
+ \frac{1 + \alpha}{1 + \beta} G(z, \bar{z}_0) \quad (26b)
\]

\[
\Psi_{Dd}, \text{ Dislocation} = -\frac{2mA}{1 + m} \left[ \frac{1 + \alpha}{1 - \beta} F(z, z_0) \right]
\]

\[
- \frac{2mA}{1 + m} \left[ \frac{1 + \alpha}{1 + \beta} G(z, z_0) \right] \quad (26c)
\]

\[
\Phi_{Dd}, \text{ Dislocation} = -2mA \left[ \frac{1 + \alpha}{1 - \beta} F(z, z_0) \right]
\]

\[
- \frac{2mA}{1 + m} \left[ \frac{1 + \alpha}{1 + \beta} F(z, \bar{z}_0) \right] \quad (26d)
\]

Each of the potentials in equation (26) contains an \(X(z)P(z)\) term following inversion. However, requiring that the stresses vanish at infinity and requiring that there be no net displacement at infinity (no net 1/z term) leads to all the \(P(z)\)'s being zero.

Removal of the constant far-field tractions is simply the standard interface problem. After performing the necessary integrals in equation (22), the \(\Omega\) potentials are:

\[
\Omega_{Dd}, \text{ Far Field} = 0 \quad (27a)
\]

\[
\Omega_{Dd}, \text{ Far Field} = \frac{\sigma_{xy}}{(1 + m)Q_1} + \frac{\sigma_{yx}}{(1 + m)Q_1} \left[ z + (2\gamma - 1)c \right] X(z)
\]

\[
+ P(z) X(z) \quad (27b)
\]

Again, \(P(z)\) is determined by far-field behavior and turns out to be zero. Inverting to standard potentials gives the fol-
Numerical Results and Conclusions

Although shear loading at infinity is included in the formulation of the problem, it is not included in the numerical results since the resulting large contact zones at the crack tips change the requirements of the solution procedure quite drastically. For this reason, the integral equation (30) was solved numerically for pure tensile load at infinity only ($\sigma_{yy}^\infty$). Although results are presented as follows for a small set of material parameters, calculations were actually carried out over the entire $\alpha, \beta$ plane, with some not unexpected numerical difficulties arising in the vicinity of extreme material mismatches. The presented results are representative of these more wide-ranging calculations.

For purposes of linking up to the asymptotic results of He and Hutchinson (1988), the parameter of interest is taken as $\mathcal{G}/\mathcal{G}_0$; the ratio of the energy release rate at the tip of the branch to the energy release rate at an equivalent unbranched interface crack. For this analysis, $\mathcal{G}$ corresponds to the strain energy release rate of a branch crack tip in material 1 and $\mathcal{G}_0$ corresponds to the strain energy release rate at the tip of an interface crack of total length $2c' = 2c + l$. The expressions for $\mathcal{G}$, $\mathcal{G}_o$, and $\mathcal{G}/\mathcal{G}_0$ are:

$$\mathcal{G} = \frac{1 - \nu_1}{2\mu_1}(K_1^2 + K_0^2)$$
$$\mathcal{G}_0 = \frac{1 + \nu_2}{\mu_1} \frac{1 - \nu_1}{\mu_2} \frac{\sigma_{yy}^\infty + \sigma_{yy}^\infty(1 + 4\epsilon^2)\pi c'}{4 \cos^2(\pi \epsilon)}$$
$$\frac{\mathcal{G}}{\mathcal{G}_0} = (1 + \alpha) \frac{\cosh^2 \pi \epsilon}{(1 + 4\epsilon^2)(\sigma_{yy}^\infty + \tau_{\infty})^2} \left( \frac{K_i}{\pi c'} \right)^2 + \left( \frac{K_{ii}}{\pi c'} \right)^2$$

As the length of the branch becomes very small relative to the main crack, it is expected that the present results should in some sense approach those of the semi-infinite crack. However, as explained below, the present results can be directly compared to He and Hutchinson’s (1988) asymptotic results only for $\beta = 0$. Figure 2 shows the dependence of the strain energy release rate, $\mathcal{G}/\mathcal{G}_0$ on the branch angle, $\theta$ for $\alpha = 0.5$, $\beta = 0$, and $1/c = 0.0001$. The $\mathcal{G}/\mathcal{G}_0$ results in Fig. 2 indeed agree well with those of He and Hutchinson (1988), as expected.

For $\beta \neq 0$, there are some interesting problems regarding the loading parameter. As shown by Rice (1988), the far-field loading and crack length are coupled in the expression for the complex intensity factor $K = K_1 + ik_2 = (k_1 + ik_3) \sqrt{\pi \cos(\pi \epsilon)}$, where $k_1 + ik_2$ is the complex intensity factor of Sih and Rice (1964). Since He and Hutchinson (1988) used $\gamma = \tan^{-1}K_{ii}/K_i$.
as their loading factor, it is not possible to match their loading parameters uniquely for \( \beta \neq 0 \). Specifically, the loading parameter used here, \( \Psi = \tan^{-1} \left( \frac{\sigma_{\text{max}}}{\sigma_{\text{y}}^0} \right) \), is related to He and Hutchinson's parameter, \( \gamma \), by:

\[
\Psi = \gamma + \tan^{-1} 2e - \log 2c. \tag{32}
\]

In the present analysis, the salient parameter is the ratio of the branch crack length to the main crack length, \( l/c \), and the absolute value of \( c \) is arbitrary. Since the value of \( c \) is arbitrary \( \Psi \) and \( \gamma \) cannot be related in a definitive manner. Equation (32) shows that for \( \beta = 0 \), \( \Psi = \gamma \) and the two loading parameters are different. As noted by Rice (1988), \( \varepsilon \) is usually very small. However, because of the \( \varepsilon \log 2c \) term in equation (32), this does not ensure that \( \Psi = \gamma \). If \( \varepsilon \log 2c < 1 \), then \( \Psi \neq \gamma \).

Figure 3 shows that for the case of the crack growing into the softer material, the maximum \( \frac{G}{G_{\text{y}}} \), \( G/G_{\text{max}} \), occurs at an angle somewhat off the interface and in the softer material \( (\theta > 0) \). Although it is not shown by these figures, the angle at which \( K_I \) is maximized in the softer material corresponds to a nearly zero value for \( K_{II} \). These results are typical for different \( l/c \) and TS(\( \Gamma = \mu_2/\mu_1 \)), and are consistent with the observations of He and Hutchinson concerning the tendency of branching to occur in the softer material provided the material toughnesses are comparable. Subsequent results are presented for \( \theta \) in the softer material only.

Figure 4 shows \( G/G_{\text{y}} \) versus \( \theta \) for various \( \Gamma \) and \( l/c = 0.001 \). Notice that as \( \Gamma \) increases, so does \( G/G_{\text{max}} \). The angle that maximizes \( G/G_{\text{y}} \), \( \theta_{\text{max}} \), also increases as \( \Gamma \) increases. Again, both these trends agree with He and Hutchinson’s (1988) results for \( \gamma \neq 0 \), although for the reasons given above, direct comparison is no longer possible. It should be noted that the in-plane stresses, \( \sigma_{\text{max}} \), are increased also by increasing material mismatch, although these stresses appear to have only secondary influence on the branch behavior.

Figures 5 and 6 show the influence of finite main crack size by considering \( G/G_{\text{y}} \) versus \( \theta \) for various \( l/c \) values. In both Figure 5 and 6, \( l/c \) increases, \( G/G_{\text{max}} \) decreases towards a value of unity, while the branch angle corresponding to maximum energy release rate decreases. (Note, however, that in calculating \( G/G_{\text{y}} \), \( G_{\text{y}} \) is adjusted to be the strain energy release rate for a crack of length \( 2c + l \). Thus, \( G_{\text{max}} \) itself is not necessarily decreasing with increasing branch length, since \( G_{\text{y}} \) is increasing with \( l \). In fact, \( G_{\text{max}} \) reaches a minimum and then starts to increase again as \( l/c \) is increased). Using \( G/G_{\text{y}} \) as a crack growth criterion, the decreasing \( \theta_{\text{max}} \) with increasing branch length can be taken as an indication that the branch would tend to turn back parallel to the interface as it grows (this rough method for predicting crack trajectories does not always work; see Rubinstein, 1989). Given the further result that the maximum energy release rate at the branch tip approaches that of the interface crack itself as the branch grows, then together this would imply that in a limited sense branching is irrelevant, since the driving force would be similar whether branching occurs or not. The critical issue, however, remains the relative toughnesses of the materials involved.

The primary conclusions to be drawn from the present analysis are that branching from an interface is a viable mode of crack growth, and that such branching is likely to occur initially at an angle somewhere between 10 deg and 47 deg in the softer of the two materials, provided the material toughnesses are comparable. As the branch extends its growth characteristics will be altered, resulting in a tendency to return to a path parallel.
to the interface, with a driving force similar to that of an unbranched crack.

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References


APPENDIX

The boundary conditions for the interface body (no crack) under remote stress are:

\[
\begin{align*}
\Omega_{31} - \Omega_{32} &= 0, \\
\Omega_{11} - \Omega_{22} &= 0
\end{align*}
\]

(A1)

As pointed out earlier, \(\Omega_{31}, \Omega_{11}\) and \(\Omega_{22}, \Omega_{22}\) are analytic in their respective regions. Moreover, since these two potentials are equal along the region boundary (as given by the above boundary condition), it can be concluded that \(\Omega_{31}\) and \(\Omega_{22}\) are analytic everywhere. By Liouville's theorem, since \(\Omega_{31}\) and \(\Omega_{22}\) are analytic everywhere and bounded, they must be constant. Thus, set:

\[
\begin{align*}
\Omega_{31}^o &= C_S - RC_S + iC_D \\
\Omega_{22}^o &= C_D - RC_D + iC_D
\end{align*}
\]

(A2a, A2b)

in which \(RC_S, IC_S, RC_D, IC_D\) and \(IC_D\) are real constants.

Inverting relations equation (A2) to Muskhelishvili (1953) potentials gives:

\[
\begin{align*}
\Phi_1^o &= Q_1 \left( \frac{1}{2\mu_2} C_S + C_D \right) \\
\Psi_1^o &= Q_1 \left( \frac{k_1}{2\mu_2} C_S + C_D \right) - Q_1 \left( \frac{1}{2\mu_2} C_S + C_D \right) \\
\Phi_2^o &= Q_2 \left( \frac{1}{2\mu_1} C_S + C_D \right) \\
\Psi_2^o &= Q_2 \left( \frac{k_2}{2\mu_1} C_S + C_D \right) - Q_2 \left( \frac{1}{2\mu_1} C_S + C_D \right)
\end{align*}
\]

(A3a, A3c)

The two complex constants \(C_S\) and \(C_D\) are determined from the conditions at infinity:

\[
\begin{align*}
\sigma_{33}^o - i\sigma_{33}^o &= \Phi_1^o + \Phi_2^o + \Psi_1^o + \Psi_2^o \\
\sigma_{11}^o - \sigma_{22}^o &= 2(\Phi_1^o + \Phi_2^o) \\
\sigma_{12}^o - \sigma_{21}^o &= 2(\Phi_1^o + \Phi_2^o)
\end{align*}
\]

(A4a, A4b, A4c)

Inverting these leads to the expressions in equation (14).